THE RECIPROCITY THEOREM FOR DEDEKIND SUMS

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1. Introduction. Let $((x)) = x - [x] - 1/2$, where $[x]$ denotes the greatest integer $\leq x$, and put

$$\bar{s}(h, k) = \sum_{r(\text{mod } k)} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right), \tag{1.1}$$

the summation extending over a complete residue system (mod k). Then if $(h, k) = 1$, the sum $\bar{s}(h, k)$ satisfies (see for example [**4**])

$$12hk\{\bar{s}(h, k) + \bar{s}(k, h)\} = h^2 + 3hk + k^2 + 1. \tag{1.2}$$

Note that $\bar{s}(h, k) = s(h, k) + 1/4$, where $s(h, k)$ is the sum defined in [**4**].

In this note we shall give a simple proof of (1.2) which was suggested by Redei's proof [**5**]. The method also applies to Apostol's extension [**1**]; [**2**].

2. A formula for $\bar{s}(h, k)$. We start with the easily proved formula

$$\left(\left(\frac{r}{k}\right)\right) = -\frac{1}{2k} + \frac{1}{k}\sum_{s=1}^{k-1} \frac{\rho^{-rs}}{\rho^s - 1} \qquad (\rho = e^{2\pi i/k}), \tag{2.1}$$

which is equivalent to a formula of Eisenstein. (Perhaps the quickest way to prove (2.1) is to observe that

$$\sum_{r=0}^{k-1} \left(\left(\frac{r}{k}\right)\right)\rho^{rs} = \begin{cases} 1/(\rho^s - 1) & (k \nmid s) \\ -1/2 & (k \mid s); \end{cases}$$

inverting leads at once to (2.1)).

Now substituting from (2.1) in (1.1) we get

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