A LATTICE-THEORETICAL FIXPOINT THEOREM AND ITS APPLICATIONS

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1. A lattice-theoretical fixpoint theorem. In this section we formulate and prove an elementary fixpoint theorem which holds in arbitrary complete lattices. In the following sections we give various applications (and extensions) of this result in the theories of simply ordered sets, real functions, Boolean algebras, as well as in general set theory and topology.[1]

By a *lattice* we understand as usual a system $\mathfrak{A} = \langle A, \leq \rangle$ formed by a non-empty set A and a binary relation $\leq$; it is assumed that $\leq$ establishes a partial order in A and that for any two elements $a,b \in A$ there is a least upper bound (join) $a \cup b$ and a greatest lower bound (meet) $a \cap b$. The relations $\geq$, $<$, and $>$ are defined in the usual way in terms of $\leq$.

The lattice $\mathfrak{A} = \langle A, \leq \rangle$ is called *complete* if every subset B of A has a least upper bound $\bigcup B$ and a greatest lower bound $\bigcap B$. Such a lattice has in particular two elements 0 and 1 defined by the formulas

$$0 = \bigcap A \quad \text{and} \quad 1 = \bigcup A .$$

Given any two elements $a,b \in A$ with $a \leq b$, we denote by $[a,b]$ the *interval* with the endpoints a and b, that is, the set of all elements $x \in A$ for which $a \leq x \leq b$; in symbols,

$$[a,b] = \mathsf{E}_x[x \in A \quad \text{and} \quad a \leq x \leq b].$$

The system $\langle [a,b], \leq \rangle$ is clearly a lattice; it is a complete if $\mathfrak{A}$ is complete.

We shall consider functions on A to A and, more generally, on a subset B of A to another subset C of A. Such a function f is called *increasing* if, for any

[1] For notions and facts concerning lattices, simply ordered systems, and Boolean algebras consult [1].

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