POWER-TYPE ENDOMORPHISMS OF SOME CLASS 2 GROUPS

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1. Introduction. Abelian groups possess endomorphisms of the form $x \longrightarrow x^n$ for each integer n. In general, however, non-abelian groups do not possess such power endomorphisms. In an earlier note, it was possible to show [**1**] for a nilpotent group G with a uniform bound on the size of the classes of conjugates that there exists an integer $n \geq 2$ for which the mapping $x \longrightarrow x^n$ is an endomorphism of G into its center. We shall consider endomorphisms of some groups of class 2 which induce power endomorphisms on the factor-commutator groups. In particular, we shall show, under suitable uniform torsion conditions for the group of inner automorphisms, that such power-type endomorphisms form a ring-like structure. Let G be a group of class 2 for which Q, the commutator subgroup, has an exponent [**2**]. Then the relation [**2**] $(xy, u) = (x, u)(y, u)$ shows that $x \longrightarrow (x, u)$ is an endomorphism of G into Q for fixed $u \in G$. Let n be any integer such that $n(n-1)/2$ is a multiple of the exponent of Q. Then the mapping $x \longrightarrow x^n(x, u)$ is a trivial example of a power-type endomorphism. If G/Q has an exponent m, we shall show that the number of distinct endomorphisms of the form $x \longrightarrow x^j$, where x^j is in the center Z of G, divides m. In particular, a non-abelian group G of class 2 has 1 or p distinct central power endomorphisms if G/Q is an elementary p-group (an abelian group with a prime p as its exponent [**2**]).

2. Power-type endomorphisms. Let G be a group with center Z and commutator subgroup Q. We assume that $Q \subset Z$ so that [**2**] G is a group of class 2. Further, suppose that there exists a least positive integer N for which $x \in G$ implies $x^N \in Z$. This means that G/Z, a group isomorphic to the group of inner automorphisms of G, is a torsion abelian group with exponent N. An endomorphism α of G will be called a *power-type* endomorphism if there exists an integer $n = n(\alpha)$ for which $\alpha(x) \equiv x^n \bmod Q$ for every $x \in G$. α induces the power endomorphism

$$\alpha^*(xQ) = x^n Q$$

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