RELATIVIZATION AND EXTENSION OF SOLUTIONS OF IRREFLEXIVE RELATIONS

MOSES RICHARDSON

1. Introduction. Let $\succ$ be an irreflexive binary relation defined over a domain $\mathfrak{D}$ of elements $a, b, c, \cdots$. We represent the system $(\mathfrak{D}, \succ)$ by an oriented graph G by regarding the elements of $\mathfrak{D}$ as vertices of G and inserting an arc ab of the graph, oriented from a to b, if and only if $a \succ b$. The sentence "$a \succ b$" is read "a dominates b". A set V of vertices is termed *internally satisfactory*[1] if and only if $x \in V$ and $y \in V$ implies $x \not\succ y$. A set V of vertices is termed *externally satisfactory* if and only if $y \in \mathfrak{D} - V$ implies that there exists an $x \in V$ such that $x \succ y$. A set V of vertices is termed a *solution* of G, or of $(\mathfrak{D}, \succ)$, if and only if it is both internally and externally satisfactory. In [4], various sufficient conditions for the existence of solutions were established.

By a *subsystem* $(\mathfrak{D}_0, \succ)$ of the system $(\mathfrak{D}, \succ)$ is meant a system where $\mathfrak{D}_0 \subset \mathfrak{D}$ and the relation $\succ$ for the subsystem is merely the restriction of the relation $\succ$ for the supersystem $(\mathfrak{D}, \succ)$. Let G_0 be the graph of the subsystem $(\mathfrak{D}_0, \succ)$ and let V_0 be a solution of G_0. A solution V of G is termed an *extension* of V_0 if $V \cap \mathfrak{D}_0 = V_0$; in this case V_0 is also said to be *relativized* from V. In this paper, some sufficient conditions for the existence of relativizations and extensions of solutions are presented. More elegant and more effective extension theorems, especially with a view toward possible applications to the theory of n-person games, remain to be desired. It is hoped that the present paper may serve to stimulate interest in this apparently difficult problem.

2. A theorem on relativization. If H is a subgraph of the graph G, then the graph obtained by adding to H all the arcs of G which join pairs of vertices of H will be termed the *juncture* of H (relative to G) and will be denoted by $\overline{H}$.

[1] In [2], internally satisfactory is called satisfactory with respect to non-domination, and in [4] it is called $\not\succ$-satisfactory.

Received March 1, 1954. Part of the work of this paper was done at the Institute for Advanced Study in 1952-3, and part while the author was consultant to the Logistics Project sponsored by the Office of Naval Research in the Department of Mathematics at Princeton University in 1953-4. A statement of many of the results contained herein appeared without proofs in [5].

Pacific J. Math. 5 (1955), 551-584