ENTIRE FUNCTIONS

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1. Introduction. The object of this note is to prove several results, which are related in that each is concerned with entire functions of exponential type. An entire function $f(z)$ is of exponential type τ if for every $\varepsilon>0$ there is a number $A=A(\varepsilon)$ such that

$$|f(z)|\leqq Ae^{(\tau+\varepsilon)|z|}. \tag{1}$$

The function is of precise type τ if (1) does not hold for any $\varepsilon<0$.

The first result is concerned with entire functions which are bounded at a sequence of points. Miss Cartwright's theorem states that if $f(z)$ is an entire function of exponential type τ, $\tau<\pi$, and is bounded by 1 at the integer points,

$$|f(n)|\leqq 1\ , \qquad n=0,\ \pm 1,\ \pm 2, \cdots, \tag{2}$$

then $f(z)$ is bounded on the entire real axis by a number which depends only on τ,

$$|f(x)|<M(\tau)\ , \qquad -\infty<x<\infty. \tag{3}$$

Proofs of this and stronger results have been given by Cartwright, Pflunger, Macintyre, Boas, Korevaar, Duffin and Schaeffer, Levin, Ahiezer, Agmon, and others. These results are discussed in [2, Chapter 10] where further references are given.

Let N be a sequence of integers. The first question to be considered in the present note is: what conditions must N satisfy in order that for every entire function of exponential type less than π the condition

$$|f(n)|\leqq 1\ , \qquad n\in N, \tag{4}$$

will imply that $f(z)$ is bounded on the real axis? To answer this question we define a function $\lambda(t)$ for $t>0$ by means of the given sequence N. Let $\lambda(t)$ be the greatest integer μ such that every interval of the real axis of length t has μ or more elements of N. For any positive t there is at least one interval, which may be supposed open, of length t which contains precisely $\lambda(t)$ elements of N, and every interval of length t whether open or closed contains $\lambda(t)$ or more elements of N. The following result is to be proved.

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