ASYMMETRY OF A PLANE CONVEX SET WITH RESPECT TO ITS CENTROID

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A. S. Besicovitch [1] proved that every bounded plane convex set K has a central subset of area at least $2m(K)/3$ where $m(K)$ denotes the area of K. His method is to construct a semi-regular hexagon of center N whose vertices belong to the boundary of K.

Ellen F. Buck and R. C. Buck [2] showed that for every K there exists at least one point X, called a six-partite point, such that there are three straight lines through X dividing K into six subsets each of area $m(K)/6$. H. G. Eggleston [3] showed that any six-partite point of K is the center of a semi-regular hexagon of area $2m(K)/3$ contained in K.

I. Fáry and L. Rédei [4] and S. Stein [5] defined for each point P the subset $S(P)$ of K determined by the intersection of K with its radial reflection in P and considered the function $f(P)=m(S(P))/m(K)$. By use of the Brunn-Minkowski theorem these authors showed that if a is a real number, then the set of points at which $f(P)\geqq a$ is convex; and the maximum f^* of $f(P)$ is attained at a single point. (Moreover, these results apply to an n-dimensional bounded convex set in n-dimensional Euclidean space.) Note that these conclusions may be false if the set K is not convex: for example, consider an L-shaped region formed by deleting one quarter of a square.

The results of Besicovitch and Eggleston imply $f(N)\geqq 2/3$ and $f(X)\geqq 2/3$, hence $f^*\geqq 2/3$.

We obtain the following theorem.

THEOREM. *If G is the centroid of K, then $f(G)\geqq 2/3$.*

To see that this result is not included in the theorems previously mentioned, consider the isosceles trapezoid with vertices $(-4, 0)$, $(4, 0)$, $(2, 2)$, $(-2, 2)$. For this example there is only one point N: $(0, 1)$ and only one point X: $(0, 4-4\sqrt{.6})$ and the closure of these points does not include G: $(0, 8/9)$.

Proof of the theorem. If K has central symmetry, then $f(G)=1$. In any case $S(G)$ has central symmetry about G; hence if K does not have central symmetry, the part M of K outside $S(G)$ has G at its centroid. Then as in Figure 1 let T be any maximal connected subset of M with

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