REMARKS ON DE LA VALLÉE POUSSIN MEANS AND CONVEX CONFORMAL MAPS OF THE CIRCLE

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Introduction. The aims of the present remarks are similar to those pursued by L. Fejér in several papers in the early nineteen thirties and well described by the title of one of his paper: *Gestaltliches über die Partialsummen und ihre Mittelwerte bei der Fourierreihe und der Potenzreihe.* However, the means which we use to realize these aims are different. Fejér discovered the remarkable behavior of certain Cesàro means, especially that of the third Cesàro means for even or odd functions of certain simple basic shapes. In what follows we show that the de la Vallée Poussin means possess such shape-preserving properties to a much higher degree thanks to their variation diminishing character.

Before stating our results, we have to explain a few concepts.

Variation diminishing Transformations on the Circle. If $a_1, a_2, \cdots, a_n$ s a finite sequence of real numbers we shall denote by $v(a)$ or $v(a_\nu)$ the number of variations of sign in the terms of this sequence. By the number $v_c(a)$ of cyclic variations of sign of our sequence we mean the following: If all $a_\nu=0$ we set $v_c(a)=0$. If $a_i\neq 0$ we set

$$v_c(a)=v(a_i, a_{i+1}, \cdots, a_n, a_1, a_2, \cdots, a_{i-1}, a_i) .$$

If we think of the a_ν as arranged clockwise in cyclic order, it becomes obvious that $v_c(a)$ does not depend on the particular non-vanishing term a_i we start with. Notice that $v_c(a)$ is always an even number. Let now $f(t)$ be a real-valued function of period 2π. Let $t_1, t_2, \cdots, t_n$ be such that

$$t_1<t_2<\cdots<t_n<t_1+2\pi . \tag{1}$$

We may now define the number $v_c(f)$ of *cyclic variations of sign of* $f(t)$ by

$$v_c(f)=\sup v_c(f(t_\nu)) , \tag{2}$$

the supremum being taken for all finite sequences $\{t_\nu\}$ subject to (1).

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