CESÀRO PARTIAL SUMS OF HARMONIC SERIES EXPANSIONS

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1. Introduction. Let the harmonic function $v(r, \theta)$ have the sine series expansion

$$v(r, \theta) = \sum_1^\infty a_\nu r^\nu \sin \nu\theta \,, \tag{1.1}$$

convergent for $0 \leqq r < 1$ and suppose that $v(r, \theta)$ is non-negative for $0 < \theta < \pi$. Denote the nth partial sum of (1.1) by

$$S_n^{(0)}(r, \theta) = \sum_1^n a_\nu r^\nu \sin \nu\theta \,, \tag{1.2}$$

and the nth Cesàro partial sum of order k, by

$$S_n^{(k)}(r, \theta) = \sum_1^n C_k^{n+k-\nu} a_\nu r^\nu \sin \nu\theta \,, \qquad k = 1, 2, \cdots . \tag{1.3}$$

It was shown by Fejér [2, p. 61] and Szász [8] that when $v(r, \theta) \geqq 0$ for $0 < \theta < \pi$, $0 < r < 1$, then $S_n^{(0)}(r, \theta)$ is also non-negative for all n when $0 < \theta < \pi$, $0 < r \leqq 1/4$, and the constant $1/4$ is sharp. Fejér [2] showed that the functions $S_n^{(3)}(1, \theta)$ are also non-negative for all n, $0 < \theta < \pi$. In addition, Szász [8] showed that there exists an $R_n^{(0)}$, depending upon n only, so that $S_n^{(0)}(r, \theta) \geqq 0$ for $0 < r \leqq R_n^{(0)}$, $0 \leqq \theta \leqq \pi$, but not always for $r > R_n^{(0)}$, and that

$$R_n^{(0)} = 1 - 3\frac{\log n}{n} + \frac{\log \log n}{n} + O(1/n) \,. \tag{1.4}$$

In this paper we shall extend the results of Szász to Cesàro partial sums of integral order k, $k = 1, 2, 3$. For $k = 3$ the theorem obtained is a sharpened form of the theorem of Fejér [2]. We prove the following:

THEOREM 1. *Let the harmonic series expansion*

$$v(r, \theta) = \sum_1^\infty a_\nu r^\nu \sin \nu\theta$$

be convergent for $0 \leqq r < 1$ *and let* $v(r, \theta) \geqq 0$ *for* $0 < \theta < \pi$, $0 < r < 1$. *Then for* $k = 0, 1, 2, 3$ *there exists a positive number* $R_n^{(k)}$ *depending upon* n *only, so that*

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