ASYMPTOTIC PERTURBATION SERIES FOR CHARACTERISTIC VALUE PROBLEMS

C. A. SWANSON

1. Introduction. Ordinary linear differential operators of the type

$$L = p_n \frac{d^n}{dx^n} + p_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_0 \qquad (n \geq 2) \tag{1.1}$$

will be under consideration on a half-open real interval $(0, b]$ $(b > 0)$, designated as the *basic interval*. The coefficients $p_j = p_j(x)(j = 0, 1, \cdots, n)$ are real-valued, continuous functions possessing j continuous derivatives on $(0, b]$, and $p_n(x) \neq 0$ on $(0, b]$. The point $x = 0$ is supposed to be a singularity for L.

The *basic operator* over the Hilbert space $\mathscr{L}^2(0, b)$ will be obtained as a restriction of L to a domain consisting of functions which are sufficiently differentiable and which satisfy certain boundary conditions. When L coincides with its Lagrangian adjoint, conditions are known [1] under which an operator like this is self-adjoint over $\mathscr{L}^2(0, b)$. Our attention will not be focused on a self-adjoint operator, however, but on a basic operator which has at least one isolated point in its spectrum.

The investigation here concerns the spectrum of a *perturbed operator*. Let $[\varepsilon, b]$ denote a closed subinterval of the basic interval, where ε is a small positive number. A *perturbed operator* A_ε is a restriction of L to a domain in $\mathscr{L}^2$ (ε, b) consisting of functions which are suitably differentiable on $[\varepsilon, b]$, and which satisfy homogeneous boundary conditions at the endpoints $x = \varepsilon$ and $x = b$. Then a set of perturbed operators is obtained when ε varies. It will be shown that for each characteristic value Λ of the basic operator, there is a characteristic value $\lambda(\varepsilon)$ of the perturbed operator A_ε which converges to Λ as $\varepsilon \to 0$; and furthermore that $\lambda(\varepsilon)$ can be represented by an asymptotic expansion, valid as $\varepsilon \to 0$.

An asymptotic expansion for the characteristic function u corresponding to $\lambda(\varepsilon)$ will also be established. In particular, the asymptotic form $u(x) = U(x)[1 + o(1)]$ will be obtained, in terms of the characteristic function U of the basic operator corresponding to Λ, valid uniformly for x contained in a certain closed subset of $[\varepsilon, b]$ as $\varepsilon \to 0$. Evidently such an asymptotic form cannot hold uniformly near the zeros of U, nor can it hold near the boundary $x = \varepsilon$ since u is forced to satisfy a boundary condition at $x = \varepsilon$. The procedure used herein permits a representation for the characteristic functions to be obtained in the "boundary layer"

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