# VARIATIONAL ASPECTS OF GENERALIZED CONVEX FUNCTIONS

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**1. Introduction.** For a second order linear homogeneous differential equation

$$L(y) \equiv y'' + p_1(x)y' + p_2(x)y = 0 , \tag{1.1}$$

with $p_1(x)$, $p_2(x)$ continuous real-valued functions on an open interval $(a, b)$ of the real line, and such that for arbitrary $x_1$, $y_1$, $x_2$, $y_2$ with $a < x_1 < x_2 < b$ there is a unique solution $y(x) = y(x;\ x_1,\ y_1;\ x_2,\ y_2)$ of (1.1) satisfying $y(x_\alpha) = y_\alpha$, $(\alpha = 1, 2)$, a real-valued function $u(x)$ has been termed "sub-$(L)$ on $(a, b)$" if for arbitrary $c$, $d$ on $a < c < d < b$ we have

$$u(x) \leqq y(x;\ c,\ u(c);\ d,\ u(d)) \ \ on\ \ c \leqq x \leqq d .$$

The class of such sub-$(L)$ functions is a special instance of sub-$F$ functions as introduced by Beckenbach [1], who established for general sub-$F$ functions various properties analogous to those of convex functions.

In particular, for sub-$(L)$ functions it has been established by Peixoto [8] and Bonsall [3] that a real-valued function $u(x)$ of class $C''$ on $(a, b)$ is sub-$(L)$ on this interval if and only if $L(u) \geqq 0$ on $(a, b)$; indeed, Peixoto has shown that for certain types of non-linear second order differential equations the corresponding sub-functions of class $C''$ are characterized by a similar differential inequality. Now if $a < x_0 < b$ and

$$r_0(x) = \exp\left[\int_{x_0}^{x} p_1(t)\, dt\right],\ \ p_0(x) = -\ p_2(x)r_0(x) ,$$

then for a function $u(x)$ of class $C''$ the condition $L(u) \geqq 0$ on $(a, b)$ is equivalent to the condition that on each compact subinterval $[c, d]$ of $(a, b)$ the function $u(x)$ affords a minimum to the integral

$$\int_c^d [r_0(x)y'^2 + p_0(x)y^2]\, dx$$

in the class of $y(x)$ that are absolutely continuous with $y'(x)$ of integrable square on $[c, d]$, and

$$y(c) = u(c),\ \ y(d) = u(d),\ \ y(x) \leqq u(x)\ \ on\ \ [c,\ d] .$$

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