ON SOME COVERING AND INTERSECTION PROPERTIES IN MINKOWSKI SPACES

B. GRÜNBAUM

1. Introduction Let X denote a Minkowski space (i.e. a finite dimensional normed linear space over the reals) and let $S = \{x \in X; ||x|| \leq 1\}$ denote the unit cell of X. In this note we shall be concerned with two numbers, E_X and J_X, determined by the geometric properties of X.

DEFINITION 1. The *expansion constant* E_X of X is the greatest lower bound of real numbers $\mu \geq 0$ which possess the following property:

Given any family $\{x_i + \alpha_i S; i \in I\}$ of mutually intersecting cells (in other words, given any family $\{x_i\}$ of points and any family $\{\alpha_i\}$ of non-negative numbers such that $||x_i - x_j|| \leq \alpha_i - \alpha_j$ for all $i, j \in I$); then

$$\bigcap_{i \in I} (x_i + \mu\alpha_i S) \neq \phi .$$

DEFINITION 2. *Jung's constant* J_X of X is the greatest lower bound of real numbers μ which possess the following property:

Given any family $\{x_i + S; i \in I\}$ of mutually intersecting cells (i.e. given any family $\{x_i\}$ such that $||x_i - x_j|| \leq 2$ for all $i, j \in I\}$; then

$$\bigcap_{i \in I} (x_i + \mu S) \neq \phi .$$

We note the following immediate consequences of the above definitions:

(i) By Helly's theorem on intersections of convex sets, the index set I may be assumed to consist of not more than $n + 1$ elements, where n is the dimension of X.

(ii) Standard compactness arguments show that E_X and J_X are not only the greatest lower bounds, but even the minima of the numbers μ defining them.

(iii) $1 \leq J_X \leq E_X \leq 2$.

(iv) J_X may equivalently be defined as the smallest number such that a cell of that diameter may cover, after a suitable translation, any set of diameter ≤ 1 .

Received November 12, 1958. The results of this paper form part of Chapter 5 of the author's Ph. D. thesis, "On some properties on Minkowski space" (in Hebrew), prepared under the guidance of Professor A. Dvoretzky at The Hebrew University in Jerusalem. The author wishes to express his sincere gratitude to Professor Dvoretzky for his helpful suggestions and criticism. The results have also been incorporated in a report on "Extensions, Retractions, and Projection", prepared in part under Contract AF 61 (052)-04.