RECURRENT MARKOV CHAINS

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In this paper Markov chains $\{X_i\}$, $i = 1, 2, \cdots$, which stationary transition probabilities are considered which take values in some measurable space $(S, \mathscr{B})$ and satisfy

(*) The Borel field $\mathscr{B}$ is separable and there exists a sigma finite measure m on $(S, \mathscr{B})$ such that P[entering E at some time $[X_0 = x] = 1$ for all $x \in S$ and all $E \in \mathscr{B}$ with $m(E) > 0$, where P is the underlying probability measure.

Such chains were introduced by Harris in [6], [7]. Let $P^n(x, E)$ be the n-step transition probability, $P^1(x, E) = P(x, E)$. In [7] it is proved that there exists a unique (up to constant factor) sigma finite measure Q which is *stationary* in the sense that $Q(E) = \int_S P(x, E)Q(dx)$.

Section 1 establishes some preliminary results. The relationship between (*) and Doeblin's condition is investigated. The results of Harris [6], [7] are summarized and extended. Note that many notational conventions used throughout the paper are introduced in § 1.

In § 2 it is shown that after the deletion of an inessential Q-null set the process splits up into a finite number, d, of disjoint cyclically moving classes.

Section 3 studies the asymptotic behavior of $P^n(x,\cdot)$ in case the stationary measure Q happens to be a probability measure. The approach is the "direct" approach of Markov and Doeblin and Doob [4]. It is shown that if $d = 1$, the total variation of $(P^n(x,\cdot) - Q)$ approaches 0 as n approaches ∞; for $d > 1$ the convergence statement must be modified in an obvious way. For the relationship of these results to those of [3] see the beginning of § 3.

Section 4 considers the asymptotic behavior of

$$U(n) = \sum_{N=1}^{\infty} P\left[\sum_{i=1}^{N} f(X_i) = n\right],$$

where f is a measurable function from S into the positive integers. If $\int_S f(x)Q(dx) < \infty$, $U(n)$ is for large n approximately a periodic function. The period depends both on the $\{X_i\}$ process and on f; this period may be greater than 1 even though the d associated with the $\{X_i\}$ process is 1 and $f(x) = 1$ for a set of x of positive Q-measure.

Section 5 is concerned with the behavior of normed sums,

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