ARCS IN PARTIALLY ORDERED SPACES

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We present here a theorem on the existence of arcs in partially ordered spaces, and several applications to topological semigroups. The hypotheses are motivated by the structure of the partially ordered set of principal ideals of a compact connected topological semigroup with unit. Noteworthy among the applications are (1)[1] a compact, connected topological semigroup with unit contains an arc. (2) A compact, connected topological semigroup with zero, each of whose elements is idempotent, is arc-wise connected. Throughout the paper, arc is used in the sense of "continuum irreducibly connected between two points". We do not assume metricity of the spaces, but all spaces are assumed to be Hausdorff. Simple non-metric examples of the theorems are furnished by the "long line", i.e. the ordinals up to and including Ω, filled in with intervals, the operation being $a \cdot b = \min(a, b)$. The author is indebted to R. D. Anderson, R. P. Hunter, and W. Strother for useful suggestions.

We recall the following definitions: [10] $(X, \leq)$ is a partially ordered space if X is a space, and $\leq$ is a reflexive, antisymmetric, transitive binary relation on X. A chain in X is an ordered subset of $(X, \leq)$. We denote by Graph $(\leq)$ the set of pairs (x, y) with $x \leq y$. We denote by $A \backslash B$ the complement of B in A; closure is denoted by*, $F(A)$ denote the boundary of A, and $\square$ denotes the empty set.

The following result of the author [3] is presented here in detail because of its relation with Theorem 2.

THEOREM 1. *Let $(X, \leq)$ be a compact partially ordered space and let W be an open set in X. If*

(1) *For each $x \in X$, $\{y \mid y \leq x\}$ is closed, and*

(2) *For any $x \in W$, each open set about x contains an element y with $y < x$,*

then if C is any component of W, $C^* \cap F(W) \neq \square$.

Proof. We show first that if V is open and $V \subset W$, then $F(V) \neq \square$. Let M be a maximal chain in V^*. Then M is compact [10], hence M has a minimal element $m \in V^*$. If $m \in V$, then by hypothesis (2) above, the chain M can be extended, contrary to the maximality. Now

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* [1]This settles a question raised by D. Montogomery. The author learned of the question through A. D. Wallace.