ON CONDITIONAL EXPECTATION AND QUASI-RINGS

M. V. JOHNS, JR. AND RONALD PYKE

1. Introduction. Let $(\Omega, \mathscr{A}, P)$ denote a complete probability space in which Ω is an arbitrary point set $(\omega \in \Omega)$, $\mathscr{A}$ is a σ-algebra of subsets of Ω $(A \in \mathscr{A})$ and P is a probability measure on $\mathscr{A}$ with respect to which P is complete. Let X, Y, Z, with or without subscripts, denote real-valued $\mathscr{A}$-measurable random variables (r. v.) Let $\mathscr{E}$ denote the space of P-integrable r. v.'s. Define a linear operator E on $\mathscr{E}$ by

$$E \circ X = \int_\Omega X \, dP .$$

E is the expectation operator and $E \circ X$ is called the expectation of X. The P-integrability criterion is equivalent to specifying $E \circ |X| < \infty$. Let $\mathscr{F}$, with or without subscripts, denote a complete σ-algebra contained in $\mathscr{A}$, and let $\mathscr{B}_k$ denote the σ-algebra of Borel sets of k-dimensional Euclidean space. For r.v.'s. $i=1, X_i, \cdots, k$, define $\mathscr{B}(X_1, \cdots, X_k) \subset \mathscr{A}$ as the minimal complete σ-algebra containing all inverse images with respect to the vector $(X_1, \cdots, X_k)$ of sets in $\mathscr{B}_k$. For $A \in \mathscr{A}$, let $I_A \in \mathscr{E}$ denote the indicator function of the set A; that is, $I_A(\omega)=1$ or 0 according as $\omega \in A$ or $\omega \notin A$. For $X \in \mathscr{E}$, define the completely-additive set function $Q_X : \mathscr{A} \to R_1$ by $Q_X(A) = E \circ XI_A$.

By the Radon-Nikodym Theorem there exists for $X \in \mathscr{E}$ and $\mathscr{F} \subset \mathscr{A}$, an $\mathscr{F}$-measurable solution $Y \in \mathscr{E}$ to the system of equations

$$E \circ (X - Y)I_A = 0 \qquad (A \in \mathscr{F}) \tag{1}$$

or equivalently

$$Q_X(A) = E \circ YI_A \qquad (A \in \mathscr{F}) .$$

This solution is unique a. s. (relative to the restriction of P to $\mathscr{F}$). The equivalence class of all such solutions (or any representative thereof) is denoted by $E\{X|\mathscr{F}\}$ and called the conditional expectation of X given $\mathscr{F}$. For $X, Y \in \mathscr{E}$ the notation $E\{X|Y\} = E\{X|\mathscr{B}(Y)\}$ will also be used. This definition of conditional expectation, which is the standard one, makes it necessary when proving theorems about conditional expectations to show at some stage of the proof that a functional equation of the form (1) is valid for all subsets of a specified σ-algebra. That this can be a tedious task is demonstrated by the existing proofs of some of the applications in § 4 of the theorems which are proved below.

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