THE TOPOLOGY OF ALMOST UNIFORM CONVERGENCE

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The theorem of Arzelà [3, 4, 6] (see Theorem 2.2) which gives a necessary condition and a sufficient condition for a net of continuous functions to converge to a continuous function plays an important part in functional analysis. In the case of linear topological spaces it has been observed that the quasi-uniform convergence [3] (see Definition 2.1) which Arzelà presented in his theorem is related to the weak and weak* topologies [6, 9, 20]. With this fact in mind it was surmised that quasi-uniform convergence would present a useful method for topologizing function spaces. This paper presents such a topology and displays some of its properties and applications. The resulting topology will be called the topology of almost uniform convergence.

In § 1 the topology is defined by means of a base for the neighborhood system of the zero function (origin). It should be noted that there is a similarity between the development of uniform convergence topologies [17] and the topology of almost uniform convergence. Section 2 shows that convergence of a net of functions for the topology implies quasi-uniform convergence. A net of functions having the property that every subnet converges quasi-uniformly will converge for the topology. In § 3 the concept of almost uniform convergence is extended to the case where convergence occurs on each member of a family of subsets of the domain space. Section 4 examines the properties of various function spaces in regard to the topology of almost uniform convergence. In particular, Theorem 4.3 shows that convergence in this topology for a net of bounded continuous functions over a regular Hausdorff space S is equivalent to pointwise convergence of their extensions on the Stone-Čech compactification of S.

Section 5 uses the topology of almost uniform convergence to obtain the weak topology for certain locally convex linear topological spaces. It is necessary in § 5 to modify the topology of almost uniform convergence to form a finer (stronger) topology which is called the topology of convex almost uniform convergence. With this new topology, Theorem 5.6 shows that the weak topology for a function space, which was originally a locally convex linear topological space for a uniform convergence topology, is the topology of convex almost uniform convergence. Theorem 5.9 parallels a theorem in Banach's book (page 134) [5] in giving a necessary and sufficient condition for the weak conver-

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