A NOTE ON KATO'S UNIQUENESS CRITERION FOR SCHRÖDINGER OPERATOR SELF-ADJOINT EXTENSIONS

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1. Introduction. Kato [2] has shown local square integrability with boundedness at ∞ of the potential coefficient function to be a sufficient condition for the Schrödinger operator in $L_2(R_n)$ to have a unique self-adjoint extension in case dimension $n = 3$. His statement is for $n = 3p$, thus with p factors R_3, but with the condition on V stated separately for each R_3 factor as is natural for application to quantum mechanics; this in essence amounts to $n = 3$ from our standpoint. Using the Young-Titchmarsh theorem on Fourier transforms, we generalize Kato's argument to general dimension $n \geq 1$. We show the connection of the resulting criterion with our earlier construction [1] of a self-adjoint extension as the inverse of a modified Green function integral operator. We also give a variational characterization of the spectrum here.

2. Uniqueness condition. Let $V(\boldsymbol{x})$ be a given, real-valued, measurable function over $\boldsymbol{x} \in R_n$, euclidean n-space. We consider the following additional conditions upon V, using the notation $(\boldsymbol{x} \cdot \boldsymbol{y}) = \sum_{j=1}^{n} x_j y_j$ and $|\boldsymbol{x}| = \sqrt{(\boldsymbol{x} \cdot \boldsymbol{x})}$ for $\boldsymbol{x}$ and $\boldsymbol{y} \in R_n$, and also denoting n dimensional Lebesque measure on R_n by μ_n.

CONDITION I. *For some* $b < +\infty$ *let* $V(\boldsymbol{x})$ *be essentially bounded* ($A = [\text{ess sup} \mid V(\boldsymbol{x}) \mid] < +\infty$) *over* $\{\boldsymbol{x} \in R_n \mid |\boldsymbol{x}| \geq b\}$, *and let*

$$(1) \qquad \int_{\{x \mid |\boldsymbol{x}| \leq b\}} |V(\boldsymbol{x})|^{(1/2)(n+\rho)} d\mu_n(\boldsymbol{x}) = M_\rho < +\infty$$

for some $\rho > 0$ *satisfying also* $n + \rho \geq 2$.

CONDITION II. *Let* $V(\boldsymbol{x})$ *satisfy Condition* I *with in addition* $n + \rho = 4$ *in* (1) *if dimension* $n < 4$.

Condition II is our generalization of Kato's uniqueness criterion, our following Theorem T. 1 in the special case $n = 3$ thus being due to Kato [2]. Following Kato, we define $\mathscr{D}_1 \subseteq L_2(R_n)$ as the linear manifold of Hermite functions, polynomials in the coordinates x_j multiplied by $\exp(-1/2 \mid \boldsymbol{x} \mid^2)$. Assuming Condition II), clearly the pointwise product $Vu \in L_2(R_n)$ for all $u \in \mathscr{D}_1$. Hence

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