ON UNIQUENESS QUESTIONS FOR HYPERBOLIC DIFFERENTIAL EQUATIONS

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1. Statement of results. This note is concerned with the existence, uniqueness, and successive approximations for solutions of the initial value problem

$$z_{xy} = f(x, y, z, p, q),\ z(x, 0) = \sigma(x),\ z(0, y) = \tau(y)\ ,$$

where $\sigma(0) = \tau(0) = z_0$, on a rectangle $R: 0 \leqq x \leqq a,\ 0 \leqq y \leqq b$. By a solution is meant a continuous function having partial derivatives almost everywhere and satisfying the integral equation

$$(1)\qquad z(x, y) = \sigma(x) + \tau(y) - z_0 + \int_0^x\int_0^y f(s, t, z(s, t), z_x(s, t), z_y(s, t))\,dsdt.$$

Actually it will be clear from the conditions imposed on σ, τ and f that any solution of (1) is uniformly Lipschitz continuous. Let D be the five-dimensional set $D = \{(x, y, z, p, q,) : (x, y) \in R$ and z, p, q arbitrary$\}$. Let $f(x, y, z, p, q)$ be defined and continuous on D, such that $|f(x, y, z, p, q,)| < N =$ const. for $(x, y, z, p, q) \in D$. Let $\sigma(x)$, $\tau(y)$ be defined and uniformly Lipschitz continuous on $0 \leqq x \leqq a$, $0 \leqq y \leqq b$, respectively (so that $|\sigma(x) - \sigma(\bar{x})| \leqq K|x - \bar{x}|$, $|\tau(y) - \tau(\bar{y})| \leqq K|y - \bar{y}|$ for some constant K) and let $\sigma(0) = \tau(0) = z_0$. In addition, for $(x, y) \in R$ and arbitrary $z, p, q, \bar{z}, \bar{p}, \bar{q}$ assume that

$$(2)\qquad |f(x, y, z, p, q,) - f(x, y, \bar{z}, \bar{p}, \bar{q})| \leqq \varphi(x, y, |z - \bar{z}|, |p - \bar{p}|, |q - \bar{q}|)\ ,$$

where $\varphi(x, y, z, p, q)$ is a continuous, non-negative function defined for $(x, y) \in R$ and non-negative z, p, q, non-decreasing in each of the variables z, p, q, and with the property that for every (α, β), where $0 < \alpha \leqq a$, $0 < \beta \leqq b$, the only solution of

$$(3)\qquad z(x, y) = \int_0^x\int_0^y \varphi(s, t, z(s, t), z_x(s, t), z_y(s, t))\,dsdt$$

in the rectangle $R_{\alpha\beta}: 0 \leqq x \leqq \alpha,\ 0 \leqq y \leqq \beta$ is $z \equiv 0$.

THEOREM (*). *Under the above assumptions on σ, τ, f and φ,* (1) *possesses one and only one solution on R. This solution is the uniform limit of the successive approximations defined by*

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