ON TERMINATING PROLONGATION PROCEDURES*

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In the classical treatments [3] of systems of differential equations there are two outstanding techniques—the Cauchy-Kowalewski theorem and completely integrable systems (the latter is really a special case of the former [1, p. 77]). In terms of systems of differential forms the Cauchy-Kowalewski theorem becomes the Cartan-Kahler theorem, and systems with independent variables which satisfy its conditions are called involutive.

Many systems are not involutive, and the central problem of prolongation theory is to construct a procedure by which one can reduce every system to an equivalent involutive system. For total prolongations Kuranishi's theorem [4, p. 44] gives a precise answer to the question of when total prolongations will lead to involutive systems. If S is the initial system in euclidean space E^n, $P^g(S)$ the g^{th} total prolongation in the space R_g, then for all points $x \in E^n$, except possibly on a proper subvariety, there is a number g_0 such that if $g \geqq g_0$ and $y \in R_g$ is a point over x, then $P^g(S)$ is involutive at y if and only if y is an ordinary integral point [4, p. 7] and the 1-forms of $P^g(S)$ do not imply any dependencies among the independent variables at integral points in a neighborhood of y. Then y is called a normal point.

The first part of this paper deals with an application of this theorem to certain types of differential systems. We show that under certain conditions the total prolongation process must result in normal points if there are to be *any* solutions. An application of this leads to a theorem often used in differential geometry [2, p. 14].

The second section is concerned with what can be done if normal points are not obtained for $P^g(S)$ as is the case with an example of Kuranishi. Here we must distinguish two cases. If $P^g(S)$ does not contain ordinary intergal points, so that its 0-forms are not a regular system of equations [4, p. 7] the Cartan-Kahler theory does not apply. Let us call such systems *singular*. We shall not consider this aspect of the problem in this paper.

If, however, the problem lies in a dependency among the independent variables implied by 1-forms of $P^g(S)$, at generic integral points, one would naturally think of restricting the system to those points where dependencies do not occur, since solutions must lie only in these points. Thus one obtains a sort of partial prolongation which could in turn be prolonged. Such a procedure was certainly what Cartan and Kuranishi had in mind. However, it is not clear that the process will ever result

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