A CLASS OF SMOOTH BUNDLES OVER A MANIFOLD

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1. Introduction. In this paper we illustrate certain constructions of importance in the geometry of smooth manifolds. First of all we prove that *a homogeneous space B of a connected Lie group G can always be represented as a homogeneous space of a contractible Lie group E*, necessarily of infinite dimension in general. In particular, that representation shows that the loop space of B can be replaced effectively by a Lie group of infinite dimension. The construction is a special case of a general theory of differentiable structures in function spaces [4]. Secondly, we examine relations between the Lie algebra of G and that of E (this latter being a Banach-Lie algebra), in case G is compact and semi-simple.

As an application we consider certain differentiable fibre bundles over a smooth (i.e., infinitely differentiable) manifold X having infinite dimensional Lie structure groups. Particular attention is given to the bundles associated with maps of X into a sphere; these bundles are important because they are in natural (Poincaré dual) correspondence with certain equivalence classes of normally framed submanifolds of X. Using a theory of smooth differential forms in function spaces, *we give explicit integral representation formulas for the characteristic classes of these bundles*. These formulas provide examples of a residue theory of differential forms with singularities [1]—and express those forms with singularities as forms without singularities in differentiable bundles over X.

2. The homogeneous spaces. (A) Let G be a connected Lie group (of finite dimension!), and let $L(G)$ denote its Lie algebra, considered as the tangent space to G at its neutral element e. If K is a closed subgroup of G, we let B denote the homogeneous space G/K of left cosets of K. The coset map $\pi : G \to B$ is an analytic fibre bundle map [9, § 7].

We now construct an *acyclic* fibre bundle over B; our construction is a variant of Serre's space of paths over B based at a point [8, Ch. IV]. For this purpose we have chosen a special class of paths on G suitable for our applications in § 5. (These path spaces are also of importance in the calculus of variations.)

(B) Let G be given a left invariant Riemann structure, determined by an inner product on $L(G)$. If $\mathscr{T}(G)$ denotes the tangent vector bundle of G with projection map $q : \mathscr{T}(G) \to G$, then $\mathscr{T}(G)$ has induced Riemann structure. If u, v are tangent vectors at a point

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