ON THE SUMMABILITY OF DERIVED FOURIER SERIES

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1. Introduction. Bosanquet ([1] and [2]) has shown that the $(C, \alpha + r)$, $\alpha \geq 0$, summability of the rth derived Fourier series of a Lebesgue integrable function $f(x)$ is equivalent to the (C, α) summability at $t = 0$ of the Fourier series of another function $\omega(t)$ (see (4), §2) integrable in the Cesaro-Lebesgue (CL) sense. This result suggests the following question: Is there a class of functions, integrable in a sense more general than that of Lebesgue, which permits such a characterization for the summability of rth derived Fourier series and which is large enough to contain $\omega(t)$ also?

In this paper it will be shown that such a characterization is possible within the class of Cesaro-Perron (CP) integrable functions for a summability scale more general than the Cesaro scale (Theorems 1 and 2, §4). Theorem 3 provides sufficient conditions for the summability of the Fourier series of $\omega(t)$ in terms of the Cesaro behavior of $\omega(t)$ at $t=0$.

Integrals are to be taken in the CP sense and of integral order, the order depending on the integrand.[1] It will be convenient to define the $C_{-1}P$ integral as the Lebesgue integral.

2. Definitions. A series Σu_ν is said to be summable (α, β) to S if

$$\lim_{n\to\infty} B \sum_{\nu<n} (1 - \nu/n)^\alpha \log^{-\beta}\left(\frac{1}{1-\nu/n}\right) u_\nu = S$$

for C sufficiently large, where $B = \log^\beta C$ and $C > 1$. (It is sufficient to say for every $C > 1$.[2])

The function $\lambda_{\alpha,\beta}(x)$ is defined by the equation:

$$\lambda_{\alpha,\beta}(x) + i\tilde{\lambda}_{\alpha,\beta}(x) = \frac{B}{\pi}\int_0^1 (1-u)^{\alpha-1} \log^{-\beta}\left(\frac{C}{1-u}\right) e^{ixu} du \,. \tag{1}$$

$$\varphi(t) \equiv \varphi(t, r, x) = \tfrac{1}{2}[f(x+t) + (-1)^r f(x-t)] \,. \tag{2}$$

$$P(t) \equiv P(t, r) = \sum_{i=0}^{[r/2]} \frac{a_{r-2i}}{(r-2i)!} t^{r-2i} \,. \tag{3}$$

$$\omega(t) = t^{-r}[\varphi(t) - P(t)] \,, \tag{4}$$

[1] Many properties of CP integration have been given by Burkill ([4], [5] and [6]) and by Sargent [7]. Other properties used in this paper can easily be verified by induction.

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[2] Bosanquet and Linfoot [3]. They have also shown the consistency of this scale for $\alpha' > \alpha$ or $\alpha' = \alpha$, $\beta' > \beta$.