CONCERNING BOUNDARY VALUE PROBLEMS[1]

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1. Introduction. This paper follows work on integral equations by H. S. Wall [4], [5], J. S. MacNerney [1], [2] and the present author [3]. Some results of these papers are used here to investigate certain boundary value problems.

In § 2, results of Wall and MacNerney are used to study a linear boundary value problem which includes problems of the following kind: Suppose that each of a_{ij}, $i, j = 1, \cdots, n$ is a continuous function, a and b are numbers and each of b_{ij}, c_{ij} and d_i, $i, j = 1, \cdots, n$ is a number. Is there a unique function n-tuple $f_1, \cdots, f_n$ such that

$$f_i' = \sum_{j=1}^{n} a_{ij} f_j \quad \text{and} \quad \sum_{j=1}^{n} [b_{ij} f_j(a) + c_{ij} f_j(b)] = d_i \,, \qquad i = 1, \cdots, n?$$

Section 3 contains some observations concerning a nonlinear boundary value problem which includes the problem of solving a certain system of nonlinear first order differential equations together with a nonlinear boundary condition. An example is given in the final section.

S denotes a normed, complete, abelian group (norms are denoted by $\|\cdot\|$). B denotes the normed, complete, abelian group of all bounded endomorphisms from S to S (the norm of an element T of B is the g.l.b. of the set of all M such that $\| Tx \| \leqq M \| x \|$ for all x in S). B^* denotes the set to which T belongs only if T is a continuous function from S to S. If $[a, b]$ denotes a number interval, then $C_{[a,b]}$ denotes the set to which f belongs only if f is a continuous function from $[a, b]$ to S. The identity function on the numbers is denoted by j.

The reader is referred to [1] for a definition of the integral of a function from a number interval $[a, b]$ to B with respect to a function from $[a, b]$ to B and to [3] for a definition of the integral of a function from $[a, b]$ to S with respect to a function from $[a, b]$ to B^*. [1] and [3] contain existence theorems for these integrals and a discussion of some of their properties.

2. A linear boundary value problem. Suppose that $[a, b]$ is a number interval and F is a continuous function from $[a, b]$ to B which is of bounded variation on $[a, b]$. The following are theorems:

(i) There is a unique continuous function M from $[a, b] \times [a, b]$ to B such that $M(t, u) = I + \int_u^t dF \cdot M(j, u)$ for each of t and u in $[a, b]$. (I denotes the identity element in B)

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