S-SPACES AND THE OPEN MAPPING THEOREM

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1. Introduction. Let E be a locally convex Hausdorff topological vector space over the field of real numbers and E' its dual. Let t_p denote the *uniform convergence* topology over precompact sets of E on E'. It is known ([2], Chapitre III, §3, N°. 5, Proposition 5) that t_p topology coincides with the weak* topology $\sigma(E', E)$ on each equicontinuous set of E'. Let t_w denote the *finest locally convex* topology on E' which coincides with $\sigma(E', E)$ on each equicontinuous set. Following H. S. Collins [3] let e-w^* denote the *finest* topology which coincides with $\sigma(E', E)$ on each equicontinuous set of E'. It is clear that these three topologies are related as follows: e-$w^* \supset t_w \supset t_p$ or, in other words, e-w^* is finer than t_w and t_w is finer than t_p. Collins [3] has shown that in general these inclusions are proper. However, if E is metrizable, e-$w^* = t_p$.

The object of this paper is to study the locally convex linear spaces E on whose dual E', e-$w^* = t_p$. Such a space we call an S-space. We shall give examples showing that an S-space is a proper generalization of metrizable locally convex linear spaces. The completion of an S-space is an S-space. A complete S-space is B-complete (a notion due to V. Pták [19] which is important in connection with the open mapping theorem). The Krein Šmulian Theorem is true on complete S-spaces (Theorem 3.) Two of the main theorems are the following ones:

THEOREM 8. *A complete l.c. space with a countable fundamental system of precompact sets is a complete S-space and hence a fortiori B-complete.*

THEOREM 10. *Let E be a complete S-space in which the closure of any dense subspace is obtained by taking the closures of its precompact sets only. Then E', endowed with the t_c-topology (the uniform convergence topology over convex compact sets of E on E') is B_r-complete for all locally convex topologies finer than t_c and coarser than $\tau(E', E)$.*

Theorem 10 is a generalization of result 6.5 [19] concerning Fréchet spaces. We also prove that an LF-space E with a defining sequence of Fréchet spaces $E_n (n \geqq 1)$ is B-complete provided each E_n contains a

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