ON THE DETERMINATION OF SETS BY THE SETS OF SUMS OF A CERTAIN ORDER

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1. Introduction. Let $X = \{x_1, \cdots, x_n\}$ be a set of (not necessarily distinct)[1] elements of a torsion free Abelian group. Define $P_s(X) = \{x_{i_1} + x_{i_2} + \cdots + x_{i_s} \mid i_1 < i_2 < \cdots < i_s\}$. Thus $P_s(X)$ has $\binom{n}{s}$ (not necessarily distinct) elements. We introduce the equivalence relation $X \sim Y$ if and only if $P_s(X) = P_s(Y)$. Let $F_s(n)$ be the greatest number of sets X which can fall into one equivalence class. Our purpose in this paper is to study conditions under which $F_s(n) > 1$. Clearly $F_s(n) = \infty$ if $n \leqq s$ so that we may restrict our attention to $n > s$.

In [5] Selfridge and Straus studied this question, restricting attention to sets of elements of a field of characteristic 0. In § 2 we show that the numbers $F_s(n)$ remain the same even if we restrict ourselves to sets of positive integers. Thus the results in [5] remain valid in our case. These included a necessary condition for $F_s(n) > 1$ and the proof that $F_2(n) > 1$ (and hence $F_{n-2}(n) > 1$) if and only if n is a power of 2. Also $F_s(2s) > 1$.

In § 3 we give a simpler form of the necessary condition in [5].

In § 4 we examine this necessary condition and prove that for $s > 2$ we have $F_s(n) = 1$ for all but a finite number of n. This was conjectured in [5]. The method seems to be of independent interest since it can be applied to a class of Diophantine equations in two unknowns which are algebraic in one and exponential in the other variable.

In § 5 we apply the methods of [5] to show that $F_2(8) = 3$, $F_2(16) \leqq 3$, $F_3(6) \leqq 6$ and $F_4(12) \leqq 2$.

The fact that $F_2(8) = 3$ disproves the conjecture $F_2(n) \leqq 2$ made in [5]. Except for the corresponding result $F_6(8) = 3$ we have not found another nontrivial case in which we can prove $F_s(n) > 2$.

In the final section we adapt a method of Lambek and Moser [3] to the case $s = 2$ and get a partial characterization of those sets which are equivalent to other sets.

2. Reduction to sets of integers. In this section we demonstrate that there exist $F_s(n)$ distinct equivalent sets of positive integers so that in any effort to evaluate $F_s(n)$ we may restrict our attention to sets of integers.

Received March 29, 1961. The work of the third author was supported in part by the National Science Foundation.

[1] Throughout this paper we use the word "set" to mean "set with multiplicities" in the sense in which one speaks of the set of zeros of a polynomial.