THE GENERALIZED WHITEHEAD PRODUCT

MARTIN ARKOWITZ

Introduction. In this paper we investigate an operation which is a generalization of the Whitehead product for homotopy groups. Let $\pi(R, S)$ denote the collection of homotopy classes of base point preserving maps of R into S, let ΣR denote the reduced suspension of R, and let $R \divideontimes S$ be the identification space $R \times S/R \vee S$ (see § 1 for complete definitions). Then this generalized Whitehead product (written GWP) assigns to each $\alpha \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$ an element $[\alpha, \beta] \in \pi(\Sigma(A \divideontimes B), X)$, where A and B are polyhedra and X is a topological space. In the case when A and B are spheres $[\alpha, \beta]$ is essentially the Whitehead product. In this paper we generalize known results on spheres and Whitehead products to polyhedra and GWPs.

The paper is divided into six parts. After the preliminaries of § 1 we present two definitions of the GWP in § 2. The first definition, which was given by Hilton in [8 ; pp. 130-131], is closely related to a commutator of group elements. The second definition is essentially a generalization of the ordinary Whitehead product. It first appeared, stated in the language of carrier theory, in a paper by D. E. Cohen [3]. We prove in Theorem 2.4 that these two definitions coincide. This generalizes a result of Samelson [11 ; p. 750].

In § 3 we establish some properties of the GWP such as anti-commutativity and bi-additivity. With the exception of Proposition 3.1 the results of this section have been obtained by Cohen [3]. However, the proofs that we give are based on the first definition and facts about commutators. Moreover, we believe that our proofs are quite elementary.

In the next section we show that $\Sigma A \times \Sigma B$ has the same homotopy type as the space obtained by attaching a cone by means of the GWP map. We then deduce a few simple consequences of this. In § 5 we consider the different ways that the GWP may be trivial. We study the following situations: (i) $[\alpha, \beta] = 0$ (ii) the GWP map is null-homotopic (iii) X is a space in which all GWPs vanish. With regard to (iii) we see that such spaces are not necessarily H-spaces.

The final section is devoted to a product which is dual (in the sense of Eckmann and Hilton) to the GWP. Two definitions of the dual product are given and they are shown to be equivalent. We also indicate some properties of the dual product.

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