THE SUM OF THE ELEMENTS OF THE POWERS OF A MATRIX

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1. Introduction and results. In the first two sections of this paper A will be assumed to be an irreducible nonnegative n-square matrix; $A \geqq 0$. Let $s_k = s_k(A)$ denote the sum of the entries in the matrix A^k, where k is a positive integer. The problem considered in the first section is the convergence of the ratio s_k/s_{k-1} as $k \to \infty$. In § 3 we obtain an inequality relating the s_k for various k in the case A is a Hermitian matrix and in § 4 we discuss convexity properties of s_2/s_1.

Let λ_1 be the dominant positive characteristic root of A which can be taken as 1 for the purposes of our subsequent arguments. If h is the number of charcteristic roots of A of modulus 1, then they are the roots of $\lambda^h - 1 = 0$ and are all simple [3]. Let $\varepsilon = e^{2\pi i/h}$ so that $1, \varepsilon, \varepsilon^2, \cdots, \varepsilon^{h-1}$ are the roots of modulus 1. Choose permutation matrices P and Q so that

$$PAP^T = \begin{bmatrix} 0 & A_1 & & & 0 \\ & 0 & A_2 & & \\ & & & \ddots & \\ & & & & A_{h-1} \\ A_h & & & & 0 \end{bmatrix} \tag{1}$$

and

$$QA^TQ^T = \begin{bmatrix} 0 & B_1 & & & 0 \\ & 0 & B_2 & & \\ & & & \ddots & \\ & & & & B_{h-1} \\ B_h & & & & 0 \end{bmatrix} \tag{2}$$

where the zero blocks down the main diagonal in both (1) and (2) are square. We shall asume henceforth that A is in this *Frobenius normal form*. In other words we assume A is already in the form given on the right in (1). Let $u_1, \cdots, u_h$ and $v_1, \cdots, v_h$ be the characteristic vectors of A and A^T corresponding to $1, \varepsilon, \cdots, \varepsilon^{h-1}$ respectively. We write for the *maximal characteristic vector*

$$u_1 = z_1 \dotplus \cdots \dotplus z_h , \tag{3}$$

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