LATTICE-ORDERED RINGS AND FUNCTION RINGS

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Introduction: This paper treats the structure of those lattice-ordered rings which are subdirect sums of totally ordered rings—the *f-rings* of Birkhoff and Pierce [4]. Broadly, it splits into two parts, concerned respectively with identical equations and with ideal structure; but there is an important overlap at the beginning.

D. G. Johnson has shown [9] that not every f-ring is *unitable*, i.e. embeddable in an f-ring which has a multiplicative unit; and he has given a characterization of unitable f-rings. We find that they form an equationally definable class. Consequently in each f-ring there is a definite l-ideal which is the obstruction to embedding in an f-ring with unit. From Johnson's results it follows that such an ideal must be nil; we find it is nilpotent of index 2, and generated by left and right annihilators.

Tarski has shown [13] that all real-closed fields are arithmetically equivalent. It follows easily that every ordered field satisfies all ring-lattice identities valid in the reals (or even in the rationals); and from a theorem of Birkhoff [2], every ordered field is therefore a homomorphic image of a latticeordered ring of real-valued functions. Adding results of Pierce [12] and Johnson [9], one gets the same conclusion for commutative f-rings which have no nonzero nilpotents. We extend the result to all zero f-rings, and all archimedean f-rings. We call these homomorphic images of f-rings of real functions *formally real* f-rings.

Birkhoff and Pierce showed [4] that f-rings themselves form an equationally definable class of abstract algebras, defined by rather simple identities involving no more than three variables. The same is true for unitable f-rings. However, no list of identities involving eight or fewer variables characterizes the formally real f-rings. The conjecture is that "eight" can be replaced by any n, but we cannot prove this.

We call an element e of an f-ring a *superunit* if $ex \geqq x$ and $xe \geqq x$ for all positive x; we call an f-ring *infinitesimal* if $x^2 \leqq |x|$ identically. A totally ordered ring is unitable if and only if it has a superunit or is infinitesimal. A general unitable f-ring is a subdirect sum of two summands, L, I, where L is a subdirect sum of totally ordered rings having superunits (we say L *has local superunits*) and I is infinitesimal. The summand L is unique.

We call a maximal l-ideal M in an f-ring A *supermodular* if A/M has a superunit. The supermodular maximal l-ideals of A, in the hull-kernel topology, form a locally compact Hausdorff space $\mathscr{M}(A)$. If A

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