ABSTRACT COMMUTATIVE IDEAL THEORY

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1. Introduction. Several years ago, M. Ward and the author [4] began a study in abstract form of the ideal theory of commutative rings. Since it was intended that the treatment should be purely ideal-theoretic, the system which was chosen for the study was a lattice with a commutative multiplication. For such multiplicative lattices, analogues of the Noether decomposition theorems for commutative rings were formulated and proved. However the theorems corresponding to the deeper results on the ideal structure of commutative rings were not obtained; the essential difficulty being the problem of formulating abstractly the notion of a principal ideal. This difficulty occurred in a mild form in treating the Noether theorem on decompositions into primary ideals. In fact, in the above mentioned paper, a weak concept of "principal element" was introduced which sufficed for the proof of the decomposition theorem into primaries. Nevertheless, the definition had serious defects and it was immediately obvious that it was not adequate for the further development of the abstract theory.

In this paper, I give a new and stronger formulation for the notion of a "principal element", and, in terms of this concept, prove an abstract version of the Krull Principal Ideal Theorem. Since there are generally many non-principal ideals of a commutative ring which are "principal elements" in the lattice of ideals, the abstract theorem represents a considerable strengthening of the classical Krull result.

It seems appropriate at this point to include a brief description of the new "principal elements" and to sketch their relationship to principal ideals.

Let L be a lattice with a multiplication and an associated residuation. The product of two lattice elements A and B will be denoted by AB and the residual, by $A:B$. An element M of L is said to be *meet principal* if

$$(1.1) \qquad (A \cap B : M)M \supseteq AM \cap B \qquad \text{all} \quad A, B \in L\,.$$

Similarly, M is said to be *join principal* if

$$(1.2) \qquad (A \cup BM) : M \subseteq A : M \cup B \qquad \text{all} \quad A, B \in L\,.$$

Finally, M is said to be *principal* if it is both meet and join principal.

Now let L be the lattice of ideals of a commutative ring R and let $M = (m)$ be a principal ideal of R. If $x \in AM \cap B$, then $x \in B$ and

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