CLOSED EXTENSIONS OF THE LAPLACE OPERATOR DETERMINED BY A GENERAL CLASS OF BOUNDARY CONDITIONS

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1. Introduction. This paper is concerned with the spectral theory of closed operators in Hilbert space determined by the Laplace operator and certain general boundary conditions. The method is that of J. W. Calkin [5, 6, 7, 8]. In this series of papers Calkin developed a theory of abstract symmetric boundary conditions in Hilbert space, and indicated how these general results might be applied to the Laplace operator and slightly more general operators on certain regions in the plane conformally equivalent to the unit circle. The boundary conditions there are of the type $\partial u/\partial n = Lu$, where L is an arbitrary, bounded, self-adjoint operator in $L_2(\partial G)$. The potential theoretic details necessary to apply the general results were given in Calkin's thesis, but not published elsewhere. They were subsequently also obtained by J. W. Smith [21] (for the case of the unit circle), who studied cases where the operator L was unbounded. R. S. Freeman [12] extended Calkin's results to a general class of plane domains and obtained a method for treating unbounded domains in E^m, $m \geqq 2$, once the results were known for bounded domains. In this paper we treat the case of a bounded domain in E^m with $C^{1,1}$ boundary. In addition, we extend the method to cover the case for which the operator L in the boundary condition is not necessarily self-adjoint. The case of unbounded domains is treated in another paper [13].

Following Calkin's method, we show there exists an appropriate linear class of functions $\mathscr{D}_1(G) \subseteqq L_2(G)$ such that the operator S in the Hilbert space $L_2(G) \oplus L_2(\partial G)$ with domain

$$\mathscr{D}(S) = \{[u, \tilde{u}] \mid u \in \mathscr{D}_1(G)\}$$

and

$$S[u, \tilde{u}] = [-\Delta u, \tilde{u}_n]\ , \quad u \in \mathscr{D}_1(G)\ ,$$

is self-adjoint. Here $\tilde{u}$ and $\tilde{u}_n$ are the values of u and $\partial u/\partial n$ on ∂G. If L is an arbitrary (not necessarily self-adjoint) bounded operator in $L_2(\partial G)$, then the operator $T_L = -\Delta$ on the domain

$$\mathscr{D}(T_L) = \{u \in \mathscr{D}_1(G) \mid \tilde{u}_n = L\tilde{u}\}$$

is closed, and T_L^* is equal to T_{L^*}. It is shown that the spectrum of T_L is discrete and is contained in a parabola with horizontal axis and opening to the right. One of our aims is a precise determination of $\mathscr{D}_1(G)$. It

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