SOME THEOREMS ON THE RATIO OF EMPIRICAL DISTRIBUTION TO THE THEORETICAL DISTRIBUTION

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1. Introduction. Let $X_1, X_2, \cdots, X_n$ be mutually independent random variables with the common cumulative distribution function $F(x)$. Let $X_1^*, X_2^*, \cdots, X_n^*$ be the same set of variables rearranged in increasing order of magnitude. In statistical language $X_1, X_2, \cdots, X_n$ form a sample of n drawn from the distribution with distribution function $F(x)$. The empirical distribution of the sample $X_1, \cdots, X_n$ is the step function $F_n(x)$ defined by

$$(1) \qquad F_n(x) = \begin{cases} 0 & \text{for } x \leqq X_1^* \\ \dfrac{k}{n} & \text{for } X_k^* < x \leqq X_{k+1}^* \\ 1 & \text{for } x > X_n^* . \end{cases}$$

A. Kolmogorov developed a well-known limit distribution law for the difference between the empirical distribution and the corresponding theoretical distribution, assuming $F(x)$ continuous:

$$\lim_{n\to\infty} P\left\{\sqrt{n} \sup_{-\infty<x<\infty} |F_n(x) - F(x)| < z\right\} = \begin{cases} 0 , & \text{for } z \leqq 0 , \\ \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2z^2} , & \text{for } z > 0 . \end{cases}$$

Equally interesting is Smirnov's theorem:

$$\lim_{n\to\infty} P\left\{\sqrt{n} \sup_{-\infty<x<\infty} [F_n(x) - F(x)] < z\right\} = \begin{cases} 0 , & \text{for } z \leqq 0 , \\ 1 - e^{-2z^2} , & \text{for } z > 0 . \end{cases}$$

In this paper we shall study the ratio of the empirical distribution to the theoretical distribution, and evaluate the distribution functions of the upper and lower bounds of the ratio. We shall prove the following four theorems.

THEOREM 1. *If F is everywhere continuous then*

$$P\left\{\sup_{0\leqq F(x)\leqq 1} \frac{F_n(x)}{F(x)} < z\right\} = s(z) = \begin{cases} 0 & \textit{for } z \leqq 1 \\ 1 - \dfrac{1}{z} & \textit{for } z > 1 . \end{cases}$$

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