ON THE RADIAL LIMITS OF BLASCHKE PRODUCTS

G. R. MacLane and F. B. Ryan

1. Introduction. As is well known, a Blaschke product $f(z)$ in $\{|z|<1\}$ has radial limits $f(e^{i\theta})$ of modulus one almost everywhere on $\{|z|=1\}$. The object of the present paper is to give a partial answer to the question: how many times does $f(z)$ assume a given radial limit? We shall prove the following theorem.

THEOREM A. *Let E be a given closed set on $\{|w|=1\}$ and let E' be the complement of E relative to $\{|w|=1\}$. Then there exists a Blaschke product $f(z)$, all of whose radial limits are of modulus one, and such that the set*

$$L(\beta)=\{\theta\,|\,f(e^{i\theta})=e^{i\beta}\}$$

has the power of the continuum for $e^{i\beta}\in E$ and is countable for $e^{i\beta}\in E'$.

Theorem A is a condensed statement of what we shall actually prove; Theorems 1, 2, and 3 contain somewhat more information on $f(z)$. The method of proof is to construct a suitable regularly-branched covering $\mathscr{W}$ of $\{|w|<1\}$, corresponding to an automorphic function $w=f(z)$, and then use the geometry of $\mathscr{W}$ to obtain our results.

The question naturally arises as to whether one could prove Theorem A directly. That is: could one produce an $f(z)$ with the desired properties by exhibiting its zeros instead of defining $f(z)$ by means of a surface $\mathscr{W}$? The answer to this question does not seem to be obvious.

2. The surface $\mathscr{W}$. Let E be a given *nonvoid* closed subset of $\{|w|=1\}$ and let $\{a_n\}_1^\infty$ be an infinite sequence of points in $\{|w|<1\}$ whose derived set is E. Clearly, we may assume that $a_n\neq 0$ and

$$\arg a_m \neq \arg a_n \qquad (m\neq n)\,. \tag{1}$$

Let $\mathscr{W}$ be the simply-connected unbordered covering of $\{|w|<1\}$ which is regularly-branched over the points $\{a_n\}$ with all branch points of multiplicity 2. It is well known [2, 3, 6] that such a covering, with any specified multiplicity or signature for each a_n, exists and is unique. Instead of appealing to the general theory of regularly-branched coverings, we shall construct the surface $\mathscr{W}$ directly, since the details of the construction play a role in the proof of Theorem A.

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