FOURIER SERIES WITH LINEARLY DEPENDENT COEFFICIENTS

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I. Introduction. The following problem is posed and solved in this article. A function $H(\theta)$ is defined over the interval $(0, \pi)$, but is as yet unknown over the interval $(-\pi, 0)$. Furthermore it is supposed that the function can be expressed as a Fourier series, with certain constraints on the coefficients. In particular

$$H(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

where

$$\alpha a_n + \beta b_n = c_n\,, \qquad n = 0, 1, 2, \cdots .$$

α and β are prescribed constants and the c_n a prescribed sequence. The question which can now be raised is whether these constraints automatically continue the function into the interval $(-\pi, 0)$. It will be shown that under certain conditions the continuation of $H(\theta)$ is unique almost everywhere.

There are two trivial special case namely if either α or β are allowed to become infinite. In these cases the proper continuation is as an odd or even function respectively.

A different, but equivalent, formulation is the following. Does the definition of $H(\theta)$ and the constraints on the Fourier coefficients a_n and b_n allow one to evaluate these coefficients? In order to be able to use the standard integral formulas for the coefficients $H(\theta)$ would have to be defined over an interval of length 2π. Over the interval $(0, \pi)$ the trigonometric functions are not orthogonal so that such integral formulas do not exist. One can show then that an equivalent statement is that the nonorthogonal set of functions $\{\sin\,(nx - \tan^{-1}\alpha/\beta)\}$ is complete in $L_2(0, \pi)$, for $|\alpha| \neq |\beta|$. The case $|\alpha| = |\beta|$ requires some additional stipulations.

One can also formulate a similar problem involving a function defined over the interval $(0, \infty)$, and constraints on the Fourier cosine and sine transforms.

In both of these case one can show that a unique continuation exists in the space of square-integrable functions for $|\alpha| \neq |\beta|$. In the case of the problem of the infinite interval one can explicitly demonstrate nonunique continuations in the space of nonintegrable functions.

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