POLYNOMIAL INTERPOLATION IN POINTS EQUIDISTRIBUTED ON THE UNIT CIRCLE

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1. Introduction. Let distinct points $S_n = \{z_{n1}, z_{n2}, \cdots, z_{nn}\}$ be given on the unit circle $|z| = 1$ in the complex z-plane, let a function f also be given on $|z| = 1$, and let $L_n = L_n(f; z)$ denote the polynomial of degree at most $n-1$ found by interpolation to f at the points S_n. Consider an infinite sequence of such point sets, $S_1, S_2, \cdots, S_n, \cdots$, and the corresponding sequence $L_1, L_2, \cdots, L_n, \cdots$. If the union of the sets S_n is everywhere dense on $|z| = 1$, does $\lim_{n\to\infty} L_n(f; z)$ exist for $|z| < 1$, and if so, what is it?

Walsh [14, pp. 178–180] proved that if the points S_n are equally spaced for each n, and if f is Riemann integrable, then

$$\lim_{n\to\infty} L_n(f; z) = \frac{1}{2\pi i}\int_{|t|=1} \frac{f(t)dt}{t-z} \tag{1.1}$$

uniformly on any closed point set on the region $|z| < 1$. The present author [1] [2] generalized Walsh's result to the case of interpolation on a more or less arbitrary Jordan curve. The problem for equally spaced interpolation points has a pedigree of some length which is described in Walsh's book [14] and in a recent survey given by the author [3].

When the points S_n are not equally spaced, very little is known about the behavior of L_n unless f is analytic on $|z| \leqq 1$. For the analytic case Fejér [4] proved that if the points S_n are equidistributed on an arbitrary Jordan curve C in a sense to be described below in § 2 and if f is analytic on the closed region $\bar{D}$ bounded by C then $L_n \to f$ uniformly on $\bar{D}$. No result of this sort involving equidistribution is at present known for nonanalytic functions f even when C is the unit circle.[1] It is the purpose of this paper to try to shed some light on the situation for nonequally spaced points by means of a probabilistic treatment. We shall let the points of the sequence $S_1, S_2, \cdots$ be random variables defined on a probability space with a structure such that almost certainly a sample sequence is equidistributed. (We use the word "equidistributed" here in connection with sample sequences rather than the more usual words "uniformly distributed" to avoid confusion with the concept of a uniform distribution in the probability sense.) The

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[1] Zygmund [**16**, vol. II, pp. 3-4] points out that a similar gap exists in the theory of trigonometric interpolation.