A NOTE ON HYPONORMAL OPERATORS

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The last exercise in reference [4] is a question to which I did not know the answer: does there exist a hyponormal ($TT^* \leqq T^*T$) completely continuous operator which is not normal? Recently Tsuyoshi Andô has answered this question in the negative, by proving that every hyponormal completely continuous operator is necessarily normal ([1]). The key to Andô's solution is a direct calculation with vectors, showing that a hyponormal operator T satisfies the relation $\| T^n \| = \| T \|^n$ for every positive integer n (for "subnormal" operators, this was observed by P. R. Halmos on page 196 of [6]). It then follows, from Gelfand's formula for spectral radius, that the spectrum of T contains a scalar μ such that $|\mu| = \| T \|$ (see [9], Theorem 1.6.3.).

The purpose of the present note is to obtain this result from another direction, via the technique of approximate proper vectors ([3]); in this approach, the nonemptiness of the spectrum of a hyponormal operator T is made to depend on the elementary case of a self-adjoint operator, and a simple calculation with proper vectors leads to a scalar μ in the spectrum of T such that $|\mu| = \| T \|$. This is the Theorem below, and its Corollaries 1 and 2 are due also to Andô. In the remaining corollaries, we note several applications to completely continuous operators.

We consider operators (=continuous linear mappings) defined in a Hilbert space. As in [3], the spectrum of an operator T is denoted $s(T)$, and the approximate point spectrum is $a(T)$. We note for future use that every boundary point of $s(T)$ belongs to $a(T)$; see, for example, ([4], hint to Exercise VIII. 3.4).

LEMMA 1. *Suppose T is a hyponormal operator, with $\| T \| \leqq 1$, and let $\mathscr{M}$ be the set of all vectors which are fixed under the operator TT^*. Then,*

(i) *$\mathscr{M}$ is a closed linear subspace,*

(ii) *the vectors in $\mathscr{M}$ are fixed under T^*T,*

(iii) *$\mathscr{M}$ is invariant under T, and*

(iv) *the restriction of T to $\mathscr{M}$ is an isometric operator in $\mathscr{M}$.*

Proof. Since $\mathscr{M} = \{x : TT^*x = x\}$ is the null space of $I - TT^*$, it is a closed linear subspace. The relation $TT^* \leqq T^*T \leqq I$ implies $0 \leqq I - T^*T \leqq I - TT^*$, and from this it is clear that the null space of $I - TT^*$ is contained in the null space of $I - T^*T$. That is, $TT^*x = x$

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