ALGEBRAIC PROPERTIES OF THE COMPOSITION OF SOLUTIONS OF PARTIAL DIFFERNTIAL EQUATIONS

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1. Introduction. Let $Lu = 0$ be a homogeneous linear partial differential equation with constant coefficients in two independent variables x and y. In [1] Lewy introduced a class of compositions each of which associates with any two solutions u and v of the differential equation a function w which depends bilinearly on u and v. The main result of [1] is that w is a solution of the same differential equation. In this paper we first give a simpler proof of this result and then investigate when the composition is commutative and associative. We prove that for every differential equation at least one of the compositions is commutative. For associativity our results are limited to second order equations. We show that when the differential equation is parabolic, one of the compositions is both commutative and associative. For elliptic and hyperbolic equations none of the compositions are associative, but by a suitable modification an operation which is both commutative and associative is obtained.

2. The compositions. The solutions u and v will be assumed to be defined in a domain $\mathscr{D}$ of the x, y-plane with the origin O as an interior point. Let x_P, y_P and x_Q, y_Q be the coordinates of the points P and Q, respectively; and let $P - Q$ be the point with coordinates $x_P - x_Q$, $y_P - y_Q$. We use ξ and η to denote operators which map functions into functions such that

$$\xi f(P) = \frac{\partial}{\partial x_P} f(P),\ \eta f(P) = \frac{\partial}{\partial y_P} f(P) . \tag{2.1}$$

Note that this implies that

$$\xi f(Q) = \frac{\partial}{\partial x_Q} f(Q),\ \eta f(Q) = \frac{\partial}{\partial y_Q} f(Q) ,$$

$$\xi f(P - Q) = \frac{\partial}{\partial x_P} f(P - Q) = -\frac{\partial}{\partial x_Q} f(P - Q) .$$

All derivatives occurring in formulas will be assumed to be continuous so that the operators ξ and η commute. When a path of integration from O to P is considered, it will always be assumed that for each

Received June 27, 1962. This research was supported by the Office of Naval Research under Contract Nonr-222 (62).