MAXIMUM MODULUS ALGEBRAS AND LOCAL APPROXIMATION IN C^n

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1. In [4] W. Rudin established an important result concerning maximum modulus algebras A of continuous complex-valued functions defined on the closure K of a Jordan domain in the complex plane (see also [5]). Rudin's result states, under the assumptions (a) A contains a function Ψ which is schlicht on K, and (b) A contains a non-constant function ϕ which is analytic in the interior, int K, of K, that every function in A is analytic in int K. In this note we will establish conditions under which assumption (b) alone yields the desired conclusion in a slightly more general setting. We assume that K is a compact set, with interior, of a Riemann surface, but also assume that int K is essentially open in the maximal ideal space Σ_A of A (A being regarded as a Banach algebra with the sup norm $\|f\| = \sup_{p \in K} |f(p)|$; see [2]). This means that each point of int K, excepting a set of points having no limit point in int K, has a neighborhood in int K which is open in Σ_A under the natural mapping of K into Σ_A. Under these assumptions it is easy to show, using the Local Maximum Modulus Principle of H. Rossi [3; Theorem 6.1] and Rudin's results, that (b) is sufficient to guarantee that A consists only of analytic functions. Our main purpose, however, is to establish the result by a geometric method, independent of Rudin's work, which is based on an appropriate local approximation in C^n. Unfortunately the geometric approach being used here only allows us to make the desired conclusion for twice continuously differentiable functions in A whereas the use of Rubin's results would give a proof valid for any function in A. However it is hoped that our method will be of some interest in itself.

The basic idea of the proof is as follows. For simplicity let K be the unit circle $\{z \in C: |z| \leqq 1\}$ in the complex plane, and let f and g be nonconstant functions in the maximum modulus algebra A. Suppose that $\Sigma_A = K$. Use f and g to map K into C^2 (the space of 2 complex variables) in the obvious way. If f and g are twice continuously differentiable in the neighborhood of a given point in int K then the image of this neighborhood in C^2 will be a two (real) dimensional surface possessing a tangent plane at the image p of the point. Let π be the two (real) dimensional tangent plane to this surface at p. If this plane is nonanalytic (Definition 1) then we can find a polynomial in the coordinates w_1 ane w_2 of C^2 which locally peaks [3] at p when

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