BANACH ALGEBRAS OF LIPSCHITZ FUNCTIONS

DONALD R. SHERBERT

1. Lip (X, d) will denote the collection of all bounded complex-valued functions defined on the metric space (X, d) that satisfy a Lipschitz condition with respect to the metric d. That is, Lip (X, d) consists of all f defined on X such that both

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

and

$$\|f\|_d = \sup\{|f(x) - f(y)|/d(x, y) : x, y \in X, x \neq y\}$$

are finite. With the norm $\|\cdot\|$ defined by $\|f\| = \|f\|_\infty + \|f\|_d$, Lip (X, d) is a Banach algebra. We shall sometimes refer to such an algebra as a Lipschitz algebra. In this paper we investigate some of the basic properties of these Banach algebras.

It will be assumed throughout the paper that (X, d) is a complete metric space. There is no loss of generality in doing so: for suppose (X, d) were not complete and let (X', d') denote its completion. Since each element of Lip (X, d) is uniformly continuous on (X, d), it extends uniquely and in a norm preserving way to an element of Lip (X', d'). Thus as Banach algebras, Lip (X, d) and Lip (X', d') are isometrically isomorphic.

In § 2 we sketch briefly the main points of the Gelfand theory and observe that every commutative semi-simple Banach algebra A is isomorphic to a subalgebra of the Lipschitz algebra Lip (Σ, σ), where Σ is the carrier space of A and σ is the metric Σ inherits from being a subset of the dual space A^* of A. This representation is obtained from the Gelfand representation; instead of using the usual Gelfand (relative weak*) topology of Σ, the metric topology is used. Later, in § 4, we show that this isomorphism is onto if and only if $A = \text{Lip}(X, d)$ for a compact (X, d).

In § 3 we study the carrier space Σ of Lip (X, d). The fact that Lip (X, d) is a point separating algebra of functions on X allows us to identify X as a subset of Σ. The topologies X inherits from Σ are compared to the original d-topology; they are shown to be equivalent and in the case of the two metric topologies we show them to be equivalent in a strong sense. In Theorem 3.9 we show that the important case of $\Sigma = X$ is equivalent to (X, d) being compact,

Received November 14, 1962. This paper is based on a portion of the author's Ph. D. dissertation written at Stanford University under the supervision of Professor Karel de Leeuw.