ON THE ELEMENTARY RENEWAL THEOREM FOR NON-IDENTICALLY DISTRIBUTED VARIABLES

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1. Introduction. Let $\{X_n\}$ be a sequence of independent, identically distributed random variables with $0 < EX_n < \infty$; write $S_n = X_1 + X_2 + \cdots + X_n$; let N_x be the number of partial sums $S_n \leq x$; write $H(x) = EN_x$. The Elementary Renewal Theorem states that under certain conditions $H(x)/x \to \{EX_n\}^{-1}$ as $x \to \infty$.

Kawata (1956) has proved a result which, as we shall see below, is equivalent to a generalization of the Elementary Renewal Theorem to the case in which the $\{X_n\}$ are non-identically distributed. Unfortunately, he found it necessary to impose quite heavy restrictions upon the distribution functions involved. In this note we shall also be concerned with the proof of the Elementary Renewal Theorem for non-identically distributed random variables, but under substantially weaker conditions than Kawata's. This renewal theorem, essentially, provides an asymptotic estimate to the sum $\sum_{n=1}^{\infty} P\{S_n \leqq x\}$; actually, we shall discuss in this paper the asymptotic behavior of more general sums $\sum_{n=1}^{\infty} a_n P\{S_n \leqq x\}$, for certain general classes of positive coefficient-sequences $\{a_n\}$. Such more general sums have also been considered by Hatori (1959), (1960), who followed Kawata's general line of attack, however, and was consequently led to assume unduly restrictive conditions.

It is well if we point out that there is another line of inquiry which could be pursued in the present context, one with which the present investigation must not be confused. Instead of considering N_x, one could define a random variable M_x as the least m for which $S_m > x$, and then study the asymptotic behavior of EM_x/x. The latter problem (also for non-identically distributed $\{X_n\}$) has been tackled in recent work announced by Robbins and Chow (1962)*. However, as might be expected, the problem we consider and the problem considered by Robbins and Chow differ in important respects, in general. Indeed, a reference to Theorem A, which we quote below, will show that one can construct a sequence of independent and *identically distributed* random variables with a finite first moment, for which EM_x is finite but EN_x is infinite. Evidently conditions which are

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* *Footnote added in proof*:—The details of this work have now appeared in Ann. Math. Statist., **34** (1963), 390–395.