PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 1, 1965

ON THE INVARIANT MEAN ON TOPOLOGICAL SEMIGROUPS AND ON TOPOLOGICAL GROUPS

EDMOND GRANIRER

Let S be a topological semigroup and $C(S)$ be the space of bounded continous functions on S. The space of translation invariant, bounded, linear functionals on $C(S)$ and its connection with the structure of S, are investigated in this paper. For topological groups G, not necessarily locally compact, the space of bounded, linear, translation invariant functionals, on the space $UC(G)$ of bounded uniformly continuous functions, is also investigated and its connection with the structure of G pointed out. The obtained results are applied to the study of the radical of the convolution algebra $UC(G)^*$ (for locally compact groups, or for subgroups of locally convex linear topological spaces) and some results which seem to be unknown even when G is taken to be the real line are obtained.

The topological semigroup S is assumed to have a separately continuous multiplication, and $C(S)$ is given the usual *sup* norm. $C(S)^*$ will denote the conjugate Banach space of $C(S)$. If $a \in S$ and f is any function on S then f_a is defined by $f_a(s) = f(as)$ for $s \in S$. $\varphi \in C(S)^*$ is said to be left invariant if $\varphi(f_a) = \varphi(f)$ for each f in $C(S)$ and a in S. $J_cl(S)$ will denote the space of left invariant elements of $C(S)^*$. A topological semigroup is said to be left amenable as a discrete semigroup if there is a linear functional $\varphi \neq 0$ on $m(S)$ (the space of all real bounded functions on S with the usual sup. norm) which satisfies $\varphi(f_a) = \varphi(f)$ for each a in S and f in $m(S)$ and $\varphi(f) \geqq 0$ if $f \geqq 0$. An analogous definition holds for the right amenable case. A topological semigroup is said to be amenable as a discrete semigroup if it is right and left amenable as a discrete semigroup.

The following are results of I. S. Luthar [12]:

(1) If S is an abelian topological semigroup with a compact ideal then $\dim J_cl(S) = 1$

(2) If G is an abelian topological group having a certain property P (Any noncompact locally compact group or any nonzero subgroup of a linear convex topological vector space has this property see [12] p. 406) then $\dim J_cl(G) \geqq 2$.

We say that a subset S_0 of the semigroup S is a left-ideal group if S_0 is a group when endowed with the multiplication induced from S

Received December 2, 1963. Supported in part by National Science Foundation Grant 19869.