PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 2, 1965

MONOTONE APPROXIMATION

O. SHISHA

How close can one approximate a monotone function by a monotone polynomial of degree $\leqq n$, or a convex function by a convex polynomial of degree $\leqq n$? This leads to the following general question. Let k and n be given, and suppose a real fuction f satisfies $f^{(k)}(x) \geqq 0$ throughout a closed, finite interval $[a, b]$. How close can one approximate f on $[a, b]$ by a polynomial of degree $\leqq n$ whose kth derivative, too, is $\geqq 0$ there? We give an answer to the question.

2. THEOREM 1. *Let k and p be integers, $1 \leqq k \leqq p$, and let a real function f satisfy throughout $[a, b]$*

$$f^{(k)}(x) \geqq 0 ,$$
$$|f^{(p)}(x_2) - f^{(p)}(x_1)| \leqq \lambda |x_2 - x_1| ,$$

λ being a constant. Then for every integer $n(\geqq p)$ there exists a real polynomial $Q_n(x)$ of degree[1] $\leqq n$ such that

(a) *$Q_n^{(k)}(x) \geqq 0$ throughout $[a, b]$,*

(b) $\displaystyle \operatorname*{Max}_{a \leqq x \leqq b} |f(x) - Q_n(x)| \leqq 2\lambda\left(\frac{\pi}{4}\right)^{p-k+1}(b-a)^{p+1}\left[k! \prod_{\nu=k}^{p} (n+1-\nu)\right]^{-1}.$

3. To prove Theorem 1, we begin by quoting the following result of J. Favard [2] and N. Ahiezer and M. Krein [1] which strengthens a previous result of D. Jackson.

THEOREM 2. (*Favard, Ahiezer-Krein*) *Let f (with period 2π) map the reals into the reals, and satisfy for every real x_1, x_2*

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1| , \tag{1}$$

λ being a constant. Then for $n = 0, 1, 2, \cdots$, there exists a trigonometric polynomial $T_n(x) \equiv \sum_{\nu=0}^{n} a_\nu^{(n)} \cos \nu x + b_\nu^{(n)} \sin \nu x$ such that $\max_{0 \leqq x \leqq 2\pi} |f(x) - T_n(x)| \leq \lambda(\pi/2)[1/(n+1)]$.

From Theorem 2 one obtains by the method of [3], pp. 13–14 the following

THEOREM 3. *Let f be a real function satisfying* (1) *throughout $[a, b]$, λ being a constant. Then for $n = 0, 1, 2, \cdots$, there exists a*

Received March 17, 1964.

[1] By degree of a polynomial we mean its exact degree. (The degree of the polynomial 0 is -1).