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THE NILPOTENT PART OF A SPECTRAL OPERATOR, II

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Let T be a spectral operator on a Banach space, such that its resolvent satisfies a mth order rate of growth condition. If N be the nilpotent part of T, it is known that $N^m = 0$ on Hilbert space. We show that $N^m = 0$ on an L_p space $(1 < p < \infty)$. Known examples show that N^m need not be zero even on an uniformly convex space.

We will consider a bounded spectral operator $T = \int \lambda E(d\lambda) + N$ which operates on an L_p space $(1 < p < \infty)$. $E(\circ)$ is the resolution of the identity and N is the nilpotent part of T [1; pp. 333–334]. We will denote by M a finite constant for which $M^{-1} \operatorname{ess}_\xi \cdot \inf \cdot | a(\xi) | \leqq \left| \int a(\xi) E(d\xi) \right| \leqq M \operatorname{ess}_\xi \cdot \sup \cdot | a(\xi) |$ is true for all bounded Borel functions $a(\xi)$, [1; Theorem 7, p. 330].

Suppose that T satisfies an mth order rate of growth condition on its resolvent: given any Borel subset σ of the spectrum of T, its restriction T_σ to the range of $E(\sigma)$ has $\bar{\sigma}$ as spectrum and we assume that for $|\zeta| \leqq |T| + 1$,

$$|(\zeta - T_\sigma)^{-1}| \leqq K[\text{distance } (\zeta, \sigma)]^{-m}$$

where K and m are constants independent of σ and ζ.

It is known that in Hilbert space, this implies $N^m = 0$ [1; Theorem 11, p. 337], and that in a reflexive Banach space $N^{m+1} = 0$, but in general no more [2; Theorem 3.1, p. 1226; Examples 4.4, p. 1230]. However, in the case of a reflexive L_p space, we will show that in fact $N^m = 0$. It is immaterial whether we show $N^m = 0$ or $N^{*m} = 0$, so that we may assume that $p \geqq 2$. We will dispense with the continual remarks that our L_p functions $x(s)$ are defined for only almost every s.

It is known that for any complex numbers $\lambda_1, \cdots, \lambda_n$ and $p \geqq 2$ we have

$$\begin{aligned}(1)\qquad \Big(\sum_{\nu=1}^n |\lambda_\nu|^2\Big)^{p/2} &\leqq (2\pi)^{-n}\int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \,|e^{i\theta_1}\lambda_1 + \cdots + e^{i\theta_n}\lambda_n|^p \\ &\leqq C(p)\Big(\sum_{\nu=0}^n |\lambda_\nu|^2\Big)^{p/2}\end{aligned}$$

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