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# A COUNTER-EXAMPLE TO A LEMMA OF SKORNJAKOV

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**In his paper, Rings with injective cyclic modules, translated in Soviet Mathematics 4 (1963), p. 36–39, L. A. Skornjakov states the following lemma: If a cyclic R-module M and all its cyclic submodules are injective, then the partially ordered set of cyclic submodules of M is a complete, complemented lattice.**

**An example is constructed to show that this lemma is false, thus invalidating Skornjakov's proof of the theorem: Let R be a ring all of whose cyclic modules are injective. Then R is semi-simple Artin. The theorem, however, is true. (See Osofsky [4].)**

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In this paper, all rings have identity and all modules are unital left modules. ${}_R\mathfrak{M}$ will denote the category of $R$-modules, and ${}_RM$ will signify $M \in {}_R\mathfrak{M}$.

Let $Q$ be a commutative, left self injective, regular, non-Artin ring, and let $I$ be a maximal ideal of $Q$ which is not a direct summand of ${}_QQ$. (For example, let $Q$ be a direct product of fields, and $I$ a maximal ideal containing their direct sum.) Let $N = Q \oplus Q/I$. We observe the following:

**1. ${}_QN$ is injective.** $Q$ is injective by hypothesis, and $Q/I$ is a simple module over the commutative regular ring $Q$; hence injective by a theorem of Kaplansky. (See [5].)

**2. ${}_QM \subseteqq {}_QN$ is a direct summand of ${}_QN$ if and only if ${}_QM$ is finitely generated.** If ${}_QM$ is a direct summand of ${}_QN$, ${}_QM$ is generated by the projections of $(1, 0 + I)$ and $(0, 1 + I)$. If ${}_QM$ is finitely generated, and $\pi$ is the projection of $N$ onto $(Q, 0 + I)$, then $\pi({}_QM)$ is finitely generated. Hence $\pi({}_QM)$ is a direct summand of ${}_QQ$. (See von Neumann [6].) Say $Q = \pi({}_QM) \oplus K$. Since $\pi({}_QM)$ is projective (it is a direct summand of $Q$), ${}_QM = (\pi({}_QM))' \oplus (\operatorname{Ker} \pi \cap {}_QM)$. Since $Q/I$ is simple, $Q/I = (\operatorname{Ker} \pi \cap {}_QM) \oplus K_2$ where $K_2 = 0$ or $Q/I$. Then $N = M \oplus K \oplus K_2$.

**3. The direct summands of $N$ do not form a lattice.** In particular, $Q(1, 0 + I) \cap Q(1, 1 + I) = (I, 0 + I)$ is not a direct summand

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