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A SUBDETERMINANT INEQUALITY

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Let A be an n-square positive semi-definite hermitian matrix and let $D_m(A)$ denote the maximum of all order m principal subdeterminants of A. In this note we prove the inequality

$$(D_m(A))^{1/m} \geqq (D_{m+1}(A))^{1/(m+1)}\ , \qquad m = 1, \cdots, n-1\ ,$$

and discuss in detail the case of equality. This result is closely related to Newton's and Szász's inequalities.

Let $A = (a_{ij})$ be an n-square positive semi-definite hermitian matrix with eigenvalues $\lambda_1 \geqq \cdots \geqq \lambda_n \geqq 0$. We introduce some notation. For $1 \leqq m \leqq n$ let $Q_{m,n}$ denote the set of all $\binom{n}{m}$ sequences $\omega = (\omega_1, \cdots, \omega_m)$, $1 \leqq \omega_1 < \omega_2 < \cdots < \omega_m \leqq n$. Let $A[\omega \mid \omega]$ denote the m-square submatrix of A whose (i, j) entry is $a_{\omega_i \omega_j}$, $i, j = 1, \cdots, m$.

THEOREM. *If A is a positive semi-definite hermitian matrix then*

$$\begin{aligned} &\max_{\alpha \in Q_{m, n}} (\det (A[\alpha \mid \alpha]))^{1/m} \\ &\qquad \geqq \max_{\omega \in Q_{m+1,n}} (\det (A[\omega \mid \omega]))^{1/m+1}, \qquad m = 1, \cdots, n-1\ . \end{aligned} \tag{1}$$

Equality holds for a given m if and only if either A has rank less than m or $A[\omega^0 \mid \omega^0]$ is a multiple of the identity, where the sequence $\omega^0 \in Q_{m+1,n}$ is one that satisfies

$$\det (A[\omega^0 \mid \omega^0]) = \max_{\omega \in Q_{m+1, n}} \det A[\omega \mid \omega]\ . \tag{2}$$

There are two classical results that are closely related to the inequalities (1). These are Szász's inequalities and the Newton inequalities. Szász proved that [1, p. 119]

$$\begin{aligned} &\left(\prod_{\alpha \in Q_{m, n}} (\det (A[\alpha \mid \alpha]))^{1/\binom{n}{m}}\right)^{1/m} \\ &\qquad \geqq \left(\prod_{\omega \in Q_{m+1,n}} (\det (A[\omega \mid \omega]))^{1/\binom{n}{m+1}}\right)^{1/(m+1)}\ . \end{aligned} \tag{3}$$

Newton's inequalities [1, p. 106] state that if $E_m(A)$ is the mth elementary symmetric function of the nonnegative numbers $\lambda_1, \cdots, \lambda_n$ then

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