PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 4, 1965

THE 2-LENGTH OF A FINITE SOLVABLE GROUP

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One measure of the structure of a finite solvable group G is its p-length $l_p(G)$. A problem connected with this measure is to obtain an upper bound for $l_p(G)$ in terms of $e_p(G)$, which is a numerical invariant of the Sylow p-subgroups of G. This problem has been solved but the best-possible result is not known for $p = 2$. The main result of this paper is that $l_2(G) \leqq 2e_2(G) - 1$, which is an improvement on earlier results. A secondary objective of this paper is to investigate finite solvable groups in which the Sylow 2-group is of exponent 4. In particular it is proved that if G is a finite group of exponent 12, then the 2-length is at most 2.

Introduction and discussion of results. The object of this paper is to obtain bounds for the 2-length of a finite solvable group. Following Hall and Higman [4], we call a finite group G p-solvable if it possesses a normal series such that each factor group is either a p-group or a p'-group. The p-length, $l_p(G)$, of such a group is the smallest number of p-groups which can occur as factor groups in such a normal series. $e_p(G)$ is defined to be the smallest n such that $x^{p^n} = 1$ for all x belonging to a Sylow p-subgroup of G.

For an odd prime p, it is proved in [4] that $l_p(G) \leqq e_p(G)$ if p is not a Fermat prime and $l_p(G) \leqq 2e_p(G)$ if p is a Fermat prime. Furthermore these results are best-possible. A. H. M. Hoare [6] then proved that in a 2-solvable group G, $l_2(G) \leqq 3e_2(G) - 2$ provided that $l_2(G) \geqq 1$. The primary purpose of this paper is to prove the following improvement:

THEOREM A. *If G is a finite solvable group and $l_2(G) \geqq 1$, then $l_2(G) \leqq 2e_2(G) - 1$.*

Feit and Thompson [1] have proved that solvability and 2-solvability are equivalent notions for finite groups. Thus no loss of generality is involved in requiring G to be solvable in the theorem.

Theorem A will be shown to be an easy consequence of the following theorem about linear groups:

THEOREM B. *Let G be a finite solvable linear group over a field*

Received July 13, 1964. This work, supported in part by a National Science Foundation Cooperative Fellowship, is based upon my doctoral dissertation at the California Institute of Technology. I wish to express my heartfelt appreciation to my advisor Marshall Hall, Jr. for his encouragement and helpful suggestions.