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CONVEXITY WITH RESPECT TO EULER-LAGRANGE DIFFERENTIAL OPERATORS

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This paper is concerned with the problem of characterizing sub-(L) functions, where L is the Euler-Lagrange operator for the functional $I_{cd}[y] = \int_c^d \left[\sum_{j=0}^{n} p_j(D^j y)^2\right]$, with n a positive integer, $[c, d]$ a subinterval of a fixed interval $[a, b]$, and $p_0, p_1, \cdots, p_n$ continuous real-valued functions on $[a, b]$ with $p_n(x) > 0$ on this interval. Under certain hypotheses on the operator L, it is shown that if f is a function in the domain of L on a subinterval $[c, d]$ of $[a, b]$, then the statement that f is sub-(L) on $[c, d]$ is equivalent to each of the following conditions: (i) $(-1)^n Lf(x) \leqq 0$ on $[c, d]$ (ii) $I_{cd}[y] \geqq I_{cd}[f]$ whenever y is a function having continuous derivatives of the first $n-1$ orders with $D^{n-1}y$ having a piecewise continuous derivative on $[c, d]$ such that $D^{j-1}y$ and $D^{j-1}f$ have the same value at c and at d for j in $\{1, \cdots, n\}$, and $y(x) - f(x) \leqq 0$ on $[c, d]$.

Section 2 is devoted to the definition and equivalent formulizations of Euler-Lagrange operators and to a discussion of adjoint operators. In § 3 it is shown that, under a hypothesis which is equivalent to the operator L being nonoscillatory on $[a, b]$, L admits a factorization of the form $(-1)^n L_0^* L_0$, where $L_0 y = \sum_{j=0}^{n} r_j D^j y$ for suitable $r_0, r_1, \cdots, r_n$. Under the further hypothesis that the operator L_0 possesses the "property W" of Polya [3], it is established that L can be written as a composition of first order real linear operators.

In § 4, the analogue of Polya's mean-value theorem in [3] is obtained for L under the above hypotheses on L and L_0. This theorem is used in §§ 5 and 6 to give characterizations, which are generalizations of results of Bonsall [1] and Reid [5] on convexity with respect to second order operators, of sub-(L) functions in terms of the operator L and the functional I_{cd}, as well as a theorem on the constancy of sign of the Green's function for a certain incompatible boundary-value problem.

Finally, in § 7, it is proved under the above assumptions on L and L_0 that the null-space of L is a $2n$-parameter family in the sense of Tornheim [7], although the relationship between sub-(L) functions and

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