PACIFIC JOURNAL OF MATHEMATICS
Vol. 16, No. 3, 1966

INFINITE PRODUCTS OF SUBSTOCHASTIC MATRICES

N. J. PULLMAN

This paper is about two types of infinite products of substochastic matrices $\{A_j\}$ namely: the left product defined by the sequence of left partial products $A_1, A_2A_1, A_3A_2A_1, \cdots$; and the right product defined by the sequence of right partial products $A_1, A_1A_2, A_1A_2A_3, \cdots$.

The basic theorem is that if the A_n are each ∞ by ∞ then:

a. There is a nonempty set E of substochastic sequences each of which (except possibly the zero sequence, 0) is the componentwise limit of a sequence of rows, one from each left partial product;

b. Any sequence $\{\rho_n\}$ of rows, one from each left partial product, can be approximated by a sequence of convex combinations $\{c_n\}$ of points of E (that is, $\{\rho_n - c_n\}$ converges componentwise to the zero sequence), and c. $E = \{0\}$ if and only if every sequence of rows, one from each left partial product, converges to 0.

Similar conclusions follow immediately for the right product of ∞ by ∞ doubly substochastic matrices.

The asymptotic behaviour of the right product of a special class of $\{A_n\}$ is also considered.

The finite case (that is, when all the A_n are r by r) for stochastic A_n is treated independently for convenience, even though the result in this case (Theorem 1) is actually a direct consequence of the basic Theorem 1′. Its conclusion is that there is an m by r stochastic matrix A with $1 \leqq m \leqq r$ and permutation matrices Q_n such that

a. if $m < r$ then for some stochastic $r - m$ by m matrices C_n:

$$\lim_{n\to\infty}\left\{A_nA_{n-1}\cdots A_1 - Q_n\begin{pmatrix}A\\ C_nA\end{pmatrix}\right\} = 0$$

and b. if $m = r$ then

$$\lim_{n\to\infty}\{A_nA_{n-1}\cdots A_1 - Q_nA\} = 0 .$$

Some results on fixed points are obtained in the finite case which carry over, in restricted form, to the infinite case.

A real matrix is said to be stochastic if none of its entries is negative and each of its row sums is 1. Two types of infinite products which arise naturally from a given sequence $\{A_n\}$ of stochastic matrices are those whose nth partial products are $R_n = A_1A_2\cdots A_n$ and $L_n = A_nA_{n-1}\cdots A_1$ respectively. We'll call the sequence $\{R_n\}$ the right

Received October 9, 1964.