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OPERATORS OF RIESZ TYPE

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The concept of an operator of Riesz type was introduced by A. F. Ruston by using as an axiomatic system those properties of compact operator used by F. Riesz in his original discussion of integral equations. In this paper we first show that this system of axioms can be somewhat simplified, and that in fact the class $\mathscr{R}$ of operators of Riesz type coincides with the class of bounded linear operators whose Fredholm region consists of all nonzero complex numbers. It is further shown that the class of strictly singular operators introduced by T. Kato and the class of inessential operators introduced by D. C. Kleinecke both lie within $\mathscr{R}$. Next, perturbation theory is considered and it is shown that with suitable commutativity conditions, $\mathscr{R}$ has the defining properties of a closed ideal. Finally, if f is analytic on an open set containing $\sigma(T)$ and $f(0) = 0$, then $f(T) \in \mathscr{R}$ if $T \in \mathscr{R}$. Moreover, if $T \in \mathscr{R}$, then the algebra generated by T also lies within $\mathscr{R}$.

Let X denote an arbitrary complex Banach space and $B(X)$ the space of bounded linear operators on X. For $T \in B(X)$, define the null manifold $N(T)$ and the range $R(T)$, also the ascent $\alpha(T)$ and the descent $\delta(T)$ as in [13]. We shall write $n(T)$ to denote the nullity of T, i.e., $\dim N(T)$, and $d(T)$ for the defect of T, i.e., the codimension of $R(T)$. Finally we define $\bar{d}(T)$, the closed defect of T, as the codimension of the uniform closure of $R(T)$. We shall not distinguish between infinite cardinals.

The five quantities $\alpha(T), \delta(T), n(T), d(T)$ and $\bar{d}(T)$ are useful in discussions of linear operators. It has been known for many years that if $\alpha(T)$ and $\delta(T)$ are finite, then they are equal. More recently, T. Kato showed [7] that if $d(T)$ is finite, then $d(T) = \bar{d}(T)$. This is clearly equivalent to concluding that $R(T)$ is closed. A more systematic attempt to relate the first four of our quantities is found in the dissertation [5] of H. Heuser. Two important relations discovered were as follows:

(1) Suppose at least one of the quantities $n(T)$ and $d(T)$ is finite. Then

(a) if $\alpha(T) < \infty$, then $d(T) \geqq n(T)$;

(b) if $\delta(T) < \infty$, then $d(T) \leqq n(T)$.

(2) Suppose $n(T) = d(T) < \infty$ and one of the quantities $\alpha(T)$, $\delta(T)$ is finite. Then the other of these two quantities is finite.