PACIFIC JOURNAL OF MATHEMATICS
Vol. 18, No. 1, 1966

THE KLEIN GROUP AS AN AUTOMORPHISM GROUP WITHOUT FIXED POINT

S. F. BAUMAN

An automorphism group V acting on a group G is said to be without fixed points if for any $g \in G$, $v(g) = g$ for all $v \in V$ implies that $g = 1$. The structure of V in this case has been shown to influence the structure of G. For example if V is cyclic of order p and G finite then John Thompson has shown that G must be nilpotent. Gorenstein and Herstein have shown that if V is cyclic of order 4 then a finite group G must be solvable of p-length 1 for all $p \mid |G|$ and G must possess a nilpotent commutator subgroup.

In this paper we will consider the case where G is finite and V noncyclic of order 4. Since V is a two group all the orbits of G under V save the identity have order a positive power of 2. Thus G is of odd order and by the work of Feit-Thompson G is solvable. We will show that G has p-lengh 1 for all $p \mid |G|$ and G must possess a nilpotent commutator subgroup.

REMARK. It would be interesting to have a direct proof of solvability without resorting to the work of Feit-Thompson.

From now on in this paper G represents a finite group admitting V as a noncyclic four group without fixed points. If X is a group admitting an automorphism group A then $Z(X)$, $\Phi(X)$, $X - A$ will be respectively the center of X, the Frattini subgroup of X and the semi-direct product of S by A in the holomorph of X. All other notations are standard.

Suppose $V = \{v_1, v_2, v_3\}$ where the v_i are the nonidentity elements of V. Denote by G_i the set of elements which are left fixed by v_i. These are easily seen to be V-invariant subgroups of G and by a result of Burnside ([1] p. 90) G_i are Abelian and v_j restricted to G_i is the inverse map if $i \neq j$. These subgroups G_i are in a sense the building blocks of G.

LEMMA 2. ([4] p. 555)

(i) $|G| = |G_1||G_2||G_3|$

(ii) $G = G_1G_2G_3$

(iii) *Every element $g \in G$ has a unique decomposition $g = g_1g_2g_3$, $f_i \in G_i$.*

LEMMA 2. *If $|G| = hm$ where $(h, m) = 1$ then G contains a unique V invariant group H such that $|H| = h$.*