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ON AN ENTIRE FUNCTION OF AN ENTIRE FUNCTION DEFINED BY DIRICHLET SERIES

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In this note we prove the following theorem which seems to exhibit an essential property of the order (R) of entire function defined by Dirichlet series.

THEOREM **If $h(s)$ and $g(s)$ are entire functions defined by Dirichlet series and $g(\log h(s))$ is an entire function of finite order (R), then there are only two possible cases: either**
(a) the internal function $h(s)$ is a Dirichlet polynomial and the external function $g(s)$ is of finite order (R); or
(b) the internal function $h(s)$ is of finite order (R) and the external function $g(s)$ is of order zero.

Here $h(s)$ and $g(s)$ are entire functions defined by the Dirichlet series

$$h(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} ,$$

$$g(s) = \sum_{n=0}^{\infty} b_n e^{ns} ,$$

satisfying the relations

$$\underset{-\infty<t<\infty}{\text{l.u.b.}} \mid h(\sigma + it) \mid = H(\sigma) ,$$

$$\underset{-\infty<t<\infty}{\text{l.u.b.}} \mid g(\sigma + it) \mid = G(\sigma) ,$$

for any real value of (in particular, every Dirichlet series absolutely convergent in the whole plane will have this property).

For this type of function Ritt [2] defines the order in the following way:

$$\rho = \limsup_{\sigma=\infty} \frac{\log \log H(\sigma)}{\sigma}$$

will be called the order (R) of $h(s)$; we shall also express it by saying that $h(s)$ is of order (R) equal to ρ.

It is to be noted that since $g(\log h(s))$ is simply a power series in $h(s)$, it is a single valued function. Since $h(s)$ is an entire function, there will be at least one term of the series which is greater than all other terms in its absolute value. In the case when there are more than one such terms we regard the term with the highest rank as the maximum term. With this convention the maximum term of $h(s)$ will be denoted by $\mu(\sigma, h)$.

2. For the proof of the above theorem we require following lemmas.