PACIFIC JOURNAL OF MATHEMATICS
Vol. 18, No. 3, 1966

EVERYWHERE DEFINED LINEAR TRANSFORMATIONS AFFILIATED WITH RINGS OF OPERATORS

ERNEST L. GRIFFIN

Let M be a ring of operators on a Hilbert space H. This paper considers conditions under which an operator T affiliated with M is bounded (or can be unbounded). In particular, we consider operators whose domain is the entire space H. We prove: THEOREM 3. If M has no type I factor part, then T is bounded. THEOREM 4. T is bounded if and only if T is bounded on each minimal projection in M. THEOREM 6. In order that every linear mapping from H into H which commutes with M be bounded, it is necessary and sufficient that M should contain no minimal projection whose range is an infinite dimensional subspace of H. These results were suggested by a theorem of J. R. Ringrose: THEOREM 8. If $M = M'$ then T is bounded.

In a paper on triangular algebras ([4], Lemma 2.12) J. R. Ringrose encountered the following situation: he was given a linear operator T with domain equal to an entire Hilbert space H and a ring of operators M commuting with T. In the case $M = M'$ (M maximal abelian) he was able to show that T had to be bounded. (For the relevant background theory, see [1, 2].) The purpose of this paper is to consider other types of rings of operators commuting with T and conditions under which T can be unbounded.

2. Since the projections in M commute with T, the ranges of these projections are invariant under T; and consequently operators are induced thereby on such subspaces. We begin by considering orthogonal families of such operators.

LEMMA 1. *If $\{E_\gamma \mid \gamma \in \Gamma\}$ is an orthogonal family of projections in M, then the norms $\{\| TE_\gamma \| \mid \gamma \in \Gamma\}$ are almost uniformly bounded; that is, there exists a finite subset Γ_0 of Γ and a positive number b such that $\| TE_\gamma \| \leqq b$ for $\gamma \in \Gamma - \Gamma_0$.*

Proof. Assume lemma false. We first choose a E_{γ_1} such that $\| TE_{\gamma_1} \| > 1$. (If $\| TE_\gamma \| \leqq 1$ for all $\gamma \in \Gamma$; then $\Gamma_0 =$ null set, $b = 1$ fulfills the lemma.) Now assume for a positive integer n that $\{E_{\gamma_k} \mid k = 1, 2, 3, \cdots, n\}$ have been chosen so that $\| TE_{\gamma_k} \| > k$ for each k. If $\| TE_\gamma \| \leqq n+1$ for $\gamma \in \Gamma - \{\gamma_k \mid k = 1, 2, 3, \cdots, n)$, then $b = n + 1$ leads to the conclusion of the lemma. Thus we can pick