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ON THE ZEROS OF A LINEAR COMBINATION OF POLYNOMIALS

ROBERT VERMES

In this paper we consider the location of the zeros of a complex polynomial $f(z)$ expressed as $f(z) = \sum_{k=0}^{n} a_k p_k(z)$ where $\{p_k(z)\}$ is a given sequence of polynomials of degree k whose zeros lie in a prescribed region E. The principal theorem states that the zeros of $f(z)$ are in the interior of a Jordan curve $S = \{z; \mid F(z) \mid = \operatorname{Max}(1, R)\}$ where F maps the complement of E onto $|z| > 1$ and R is the positive root of the equation $\sum_{k=0}^{n-1} \lambda_k \mid a_k \mid t^k - \lambda n \mid a_n \mid t^n = 0$, with $\lambda_k > 0$ depending on E only. Several applications of this theorem are given. For example; if $\{p_k(z)\}$ is a sequence of orthogonal polynomials on $a \leqq z \leqq b$, then we give an ellipse containing all the zeros of $\sum_{k=0}^{n} a_k p_k(z)$.

Previous results. An extensive mathematical literature deals with the location of the zeros in the complex plane of a polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \tag{1}$$

with complex coefficients a_j. Cauchy derived practical bounds for the moduli of the zeros of (1) using the moduli of the coefficients a_j. In many investigations the polynomial (1) is not expressed as a linear combination of the sequence $\{z^k\}$, but as

$$f(z) = b_0 + b_1 p_1(z) + \cdots + b_n p_n(z) \tag{2}$$

where $\{p_k(z)\}$ is a given sequence of polynomials. Cauchy's well known result (Marden [2], Th. 27, 1) was generalized by Turán [4] in the case where the expansion in (2) is the Hermite-expansion $e^{z^2} \sum_{k=0}^{n} b_k (e^{-z^2})^{(k)}$. He obtained upper bounds for the moduli of the imaginary parts of the zeros, i.e., a "strip" where all the zeros of (2) are located. Specht [3], making use of the Christoffel-Darboux formula, extended these results to other sequences of orthogonal polynomials. In our Theorem 1, we replace the "strip" with a bounded region, which will yield an ellipse in the case where the $\{p_k(z)\}$ is a sequence of orthogonal polynomials on a finite interval.

2. Cauchy type estimate. In the sequal we shall use the following notations: Let E be a compact (infinite) set in the complex z-plane, whose complement G is simply connected, $w = F(z)$ the univalent function which is defined on G and maps G conformally on D: $|w| > 1$ such that the point at infinity in the two planes correspond to each